Chapter 1

First-Order Differential Equations

1.1 Terminology and Separable Equations

1. For $x > 1$,
$$2\varphi\varphi' = 2\sqrt{x-1}\frac{1}{2\sqrt{x-1}} = 1,$$
so φ is a solution.

2. With $\varphi(x) = Ce^{-x}$,
$$\varphi' + \varphi = -Ce^{-x} + Ce^{-x} = 0,$$
so φ is a solution.

3. For $x > 0$, rewrite the equation as
$$2xy' + 2y = e^x.$$
With $y = \varphi(x) = \frac{1}{2}x^{-1}(C - e^x)$, compute
$$y' = \frac{1}{2}\left(-x^{-2}(C - e^x) - x^{-1}e^x\right).$$
Then
$$2xy' + 2y = x\left(-x^{-2}(C - e^x) - x^{-1}e^x\right) + x^{-1}(C - e^x) = e^x.$$
Therefore $\varphi(x)$ is a solution.

4. For $x \neq \pm\sqrt{2}$,
$$\varphi' = \frac{-2cx}{(x^2-2)^2} = \left(\frac{2x}{2-x^2}\right)\left(\frac{c}{x^2-2}\right) = \frac{2x\varphi}{2-x^2},$$
so φ is a solution.

5. On any interval not containing $x = 0$ we have

$$x\varphi' = x\left(\frac{1}{2} + \frac{3}{2x^2}\right) = x + \left(\frac{3}{2x} - \frac{x}{2}\right) = x - \left(\frac{x^2-3}{2x}\right) = x - \varphi,$$

so φ is a solution.

6. For all x,

$$\varphi' + \varphi = -Ce^{-x} + (1 + Ce^{-x}) = 1$$

so $\varphi(x) = 1 + Ce^{-x}$ is a solution.

7. Write

$$3\frac{dy}{dx} = \frac{4x}{y^2}$$

and separate variables:

$$3y^2\,dy = 4x\,dx.$$

Integrate to obtain

$$y^3 = 2x^2 + k,$$

which implicitly defines the general solution. We can also write

$$y = \left(2x^2 + k\right)^{1/3}.$$

8. Write the differential equation as

$$x\frac{dy}{dx} = -y$$

and separate the variables:

$$\frac{1}{y}\,dy = -\frac{1}{x}\,dx.$$

This separation requires that $x \neq 0$ and $y \neq 0$. Integration gives us $\ln|y| = -\ln|x| + c$. Then

$$\ln|y| + \ln|x| = c$$

so $\ln|xy| = c$. Then $xy = e^c = k$, in which k can be any positive constant. Notice now that $y = 0$ is also a solution of the original differential equation. Therefore, if we allow k to be any constant (positive, negative or zero), we can omit the absolute values and write the general solution in the implicit form $xy = k$.

9. Write the differential equation as

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin(x+y)}{\cos(y)} \\ &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(y)} \\ &= \sin(x) + \cos(x)\frac{\sin(y)}{\cos(y)}.\end{aligned}$$

There is no way to separate the variables in this equation, so the differential equation is not separable.

10. Since $e^{x+y} = e^x e^y$, we can write the differential equation as
$$e^x e^y \frac{dy}{dx} = 3x$$
or, in separated form,
$$e^y\,dy = 3xe^{-x}\,dx.$$
Integration gives us the implicitly defined general solution
$$e^y = -3e^{-x}(x+1) + c.$$

11. Write the differential equation as
$$x\frac{dy}{dx} = y(y-1).$$
This is separable. If $y \neq 0$ and $y \neq 1$, we can write
$$\frac{1}{x}\,dx = \frac{1}{y(y-1)}\,dy.$$
Use partial fractions to write this as
$$\frac{1}{x}\,dx = \frac{1}{y-1}\,dy - \frac{1}{y}\,dy.$$
Integrate to obtain
$$\ln|x| = \ln|y-1| - \ln|y| + c,$$
or
$$\ln|x| = \ln\left|\frac{y-1}{y}\right| + c.$$
This can be solved for x to obtain the general solution
$$y = \frac{1}{1-kx}.$$
The trivial solution $y(x) = 0$ is a singular solution, as is the constant solution $y(x) = 1$. We assumed that $y \neq 0, 1$ in the algebra of separating the variables.

12. This equation is not separable.

13. This equation is separable since we can write it as
$$\frac{\sin(y)}{\cos(y)}\,dy = \frac{1}{x}\,dx$$
if $\cos(y) \neq 0$ and $x \neq 0$. A routine integration gives the implicitly defined general solution $\sec(y) = kx$. Now $\cos(y) = 0$ if $y = (2n+1)\pi/2$ for n any integer. $y = (2n+1)\pi/2$ also satisfies the original differential equation and is a singular solution.

14. The differential equation itself assumes that $y \neq 0$ and $x \neq -1$. Write

$$\frac{x}{y}\frac{dy}{dx} = \frac{2y^2+1}{x+1},$$

which separates as

$$\frac{1}{y(2y^2+1)}\,dy = \frac{1}{x(x+1)}\,dx.$$

Use a partial fractions decomposition to write

$$\left(\frac{1}{y} - \frac{2y}{1+2y^2}\right)dy = \left(\frac{1}{x} - \frac{1}{1+x}\right)dx.$$

Integration this equation to obtain

$$\ln|y| - \frac{1}{2}\ln(1+2y^2) = \ln|x| - \ln|x+1| + c.$$

Then,

$$\ln\left(\frac{y}{\sqrt{1+2y^2}}\right) = \ln\left(\frac{x}{x+1}\right) + c,$$

in which we have taken the case that $y > 0$ and $x > 0$ to drop the absolute values. Finally, take the exponential of both sides of this equation to obtain the implicitly defined solution

$$\frac{y}{\sqrt{1+2y^2}} = k\left(\frac{x}{x+1}\right).$$

Since $y = 0$ satisfies the original differential equation, $y = 0$ is a singular solution.

15. This differential equation is not separable.

16. Substitute

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y),$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

and

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

into the differential equation to obtain the separated equation

$$(\cos(y) - \sin(y))\,dy = (\cos(x) - \sin(x))\,dx.$$

Upon integrating we obtain the implicitly defined solution

$$\cos(y) + \sin(y) = \cos(x) + \sin(x) + c.$$

17. If $y \neq -1$ and $x \neq 0$, we obtain the separated equation

$$\frac{y^2}{y+1}\,dy = \frac{1}{x}\,dx.$$

Write this as

$$\left(y - 1 + \frac{1}{1+y}\right)\,dy = \frac{1}{x}\,dx.$$

Integrate to obtain

$$\frac{1}{2}y^2 - y + \ln|1+y| = \ln|x| + c.$$

Now use the initial condition $y(3e^2) = 2$ to obtain

$$2 - 2 + \ln(3) = \ln(3) + 2 + c$$

so $c = -2$ and the solution is implicitly defined by

$$\frac{1}{2}y^2 - y + \ln(1+y) = \ln(x) - 2,$$

in which the absolute values have been removed because the initial condition puts the solution in a part of the $x, y-$ plane where $x > 0$ and $y > -1$.

18. Integrate

$$\frac{1}{y+2}\,dy = 3x^2\,dx$$

to obtain $\ln|2+y| = x^3 + c$. Substitute the initial condition to obtain $c = \ln(10) - 8$. The solution is defined by

$$\ln\left(\frac{2+y}{10}\right) = x^3 - 8.$$

19. Write $\ln(y^x) = x\ln(y)$ and separate the variables to write

$$\frac{\ln(y)}{y}\,dy = 3x\,dx.$$

Integrate to obtain $(\ln(y))^2 = 3x^2 + c$. Substitute the initial condition to obtain $c = -3$, so the solution is implicitly defined by $(\ln(y))^2 = 3x^2 - 3$.

20. Write $e^{x-y^2} = e^x e^{-y^2}$ and Separate the variables to obtain

$$2ye^{y^2}\,dy = e^x\,dx.$$

Integrate to get $e^{y^2} = e^x + c$. The condition $y(4) = -2$ requires that $c = 0$, so the solution is defined implicitly by $e^{y^2} = e^x$, or $x = y^2$. Since $y(4) = -2$, the explicit solution is $y = -\sqrt{x}$.

21. Separate the variables to obtain

$$y\cos(3y)\,dy = 2x\,dx,$$

with solution given implicitly by

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 + c.$$

The initial condition requires that

$$\frac{\pi}{9}\sin(\pi) + \frac{1}{9}\cos(\pi) = \frac{4}{9} + c,$$

so $c = -5/9$. The solution is implicitly defined by

$$3y\sin(3y) + \cos(3y) = 9x^2 - 5.$$

22. By Newton's law of cooling the temperature function $T(t)$ satisfies $T'(t) = k(T-60)$, with k a constant of proportionality to be determined, and with $T(0) = 90$ and $T(10) = 88$. This is based on the object being placed in the environment at time zero. This differential equation is separable (as in the text) and we solve it subject to $T(0) = 90$ to obtain $T(t) = 60 + 30e^{kt}$. Now

$$T(10) = 88 = 60 + 30e^{10k}$$

gives us $e^{10k} = 14/15$. Then

$$k = \frac{1}{10}\ln\left(\frac{14}{15}\right) \approx -6.899287(10^{-3}).$$

Since $e^{10k} = 14/15$, we can write

$$T(t) = 60 + 30(e^{10k})^{t/10} = 60 + 30\left(\frac{14}{15}\right)^{t/10}.$$

Now

$$T(20) = 60 + 30\left(\frac{14}{15}\right)^{2} \approx 86.13$$

degrees Fahrenheit. To reach 65 degrees, solve

$$65 = 60 + 30\left(\frac{14}{15}\right)^{t/10}$$

to obtain

$$t = \frac{10\ln(1/6)}{\ln(14/15)} \approx 259.7$$

minutes.

23. Suppose the thermometer was removed from the house at time $t = 0$, and let $t > 0$ denote the time in minutes since then. The house is kept at 70 degrees F. Let A denote the unknown outside ambient temperature, which is assumed constant. The temperature of the thermometer at time t is modeled by

$$T'(t) = k(T - A); T(0) = 70, T(5) = 60 \text{ and } T(15) = 50.4.$$

There are three conditions because we must find k and then A.

Separation of variables and the initial condition $T(0) = 70$ yield the expression $T(t) = A + (70 - A)e^{kt}$. The other two conditions now give us

$$T(5) = 60 = A + (70 - A)e^{5k} \text{ and } T(15) = 50.4 = A + (70 - A)e^{15k}.$$

Solve the first equation to obtain

$$e^{5k} = \frac{60 - A}{70 - A}.$$

Substitute this into the second equation to obtain

$$(7 - A)\left(\frac{60 - A}{70 - A}\right)^3 = 50.4 - A.$$

This yields the quadratic equation

$$10.4A^2 - 1156A + 30960 = 0$$

with roots $A = 45$ and 66.16. Clearly we require that $A < 50.4$, so $A = 45$ degrees Fahrenheit.

24. The amount $A(t)$ of radioactive material at time t is modeled by

$$A'(t) = kA; A(0) = e^3$$

together with the condition $A(\ln(2)) = e^3/2$, since we must also find k. Time is in weeks. Solve to obtain

$$A(t) = \left(\frac{1}{2}\right)^{t/\ln(2)} e^3$$

tons. Then $A(3) = e^3(1/2)^{3/\ln(2)} = 1$ ton.

25. Similar to Problem 24, we find that the amount of Uranium-235 at time t is

$$U(t) = 10\left(\frac{1}{2}\right)^{t/(4.5(10^9))},$$

with t in years. Then $U(10^9) = 10(1/2)^{1/4.5} \approx 8.57$ kg.

26. At any time t there will be $A(t) = 12e^{kt}$ gms, and $A(4) = 9.1$ requires that $e^{4k} = 9.1/12$, so

$$k = \frac{1}{4}\ln\left(\frac{9.1}{12}\right) \approx -0.06915805.$$

The half-life is the time t^* so that $A(t^*) = 6$, or $e^{kt^*} = 1/2$. This gives $t^* = -\ln(2)/k \approx 10.02$ minutes.

27. Compute

$$I'(x) = -\int_0^\infty \frac{2x}{t} e^{-(t^2+(x/t)^2)}\, dt.$$

Let $u = x/t$ to obtain

$$\begin{aligned} I'(x) &= 2\int_\infty^0 e^{-((x/u)^2+u^2)}\, du \\ &= -2\int_0^\infty e^{-(u^2+(x/u)^2)}\, du = -2I(x). \end{aligned}$$

This is the separable equation $I' = -2I$. Write this as

$$\frac{1}{I}\, dI = -2\, dx$$

and integrate to obtain $I(x) = ce^{-2x}$. Now

$$I(0) = \int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2},$$

a standard result often used in statistics. Then

$$I(x) = \frac{\sqrt{\pi}}{2} e^{-2x}.$$

Put $x = 3$ to obtain

$$\int_0^\infty e^{-t^2-(9/t^2)}\, dt = \frac{\sqrt{\pi}}{2} e^{-6}.$$

28. (a) For water h feet deep in the cylindrical hot tub, $V = 25\pi h$, so

$$25\pi \frac{dh}{dt} = -0.6\pi\left(\frac{5}{16}\right)^2 \sqrt{64h},$$

with $h(0) = 4$. Thus

$$\frac{dh}{dt} = -\frac{3\sqrt{h}}{160}.$$

(b) The time it will take to drain the tank is

$$T = \int_4^0 \left(\frac{dt}{dh}\right) dh$$
$$= \int_4^0 -\frac{160}{3\sqrt{h}}\, dh = \frac{640}{3}$$

seconds.

(c) To drain the upper half will require

$$T_1 = \int_4^2 -\frac{160}{3\sqrt{h}}\, dh = \frac{320}{3}(2-\sqrt{2})$$

seconds, approximately 62.5 seconds. The lower half requires

$$T_2 = \int_2^0 -\frac{160}{3\sqrt{h}}\, dh = \frac{320}{3}\sqrt{2}$$

seconds, about 150.8 seconds.

29. Model the problem using Torricelli's law and the geometry of the hemispherical tank. Let $h(t)$ be the depth of the liquid at time t, $r(t)$ the radius of the top surface of the draining liquid, and $V(t)$ the volume in the container (See Figure 1.1). Then

$$\frac{dV}{dt} = -kA\sqrt{2gh} \text{ and } \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$

Here $r^2 + h^2 = 18^2$, since the radius of the tub is 18. We are given $k = 0.8$ and $A = \pi(1/4)^2 = \pi/16$ is the area of the drain hole. With $g = 32$ feet per second per second, we obtain the initial value problem

$$\pi(324 - h^2)\frac{dh}{dt} = 0.4\pi\sqrt{h}; h(0) = 18.$$

This is a separable differential equation with the general solution

$$1620\sqrt{h} - h^{5/2} = -t + k.$$

Then $h(0) = 18$ yields $k = 3888\sqrt{2}$, so

$$1620\sqrt{h} - h^{5/2} = 3888\sqrt{2} - t.$$

The hemisphere is emptied at the instant that $h = 0$, hence at $t = 3888\sqrt{2}$ seconds, about 91 minutes, 39 seconds.

30. From the geometry of the sphere (Figure 1.2), $dV/dt = -kA\sqrt{2gh}$ becomes

$$\pi(32A - (h-18)^2)\frac{dh}{dt} = -0.8\pi\left(\frac{1}{4}\right)^2\sqrt{64h},$$

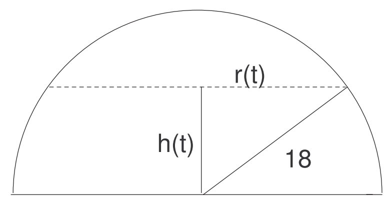

Figure 1.1: Problem 29, Section 1.1.

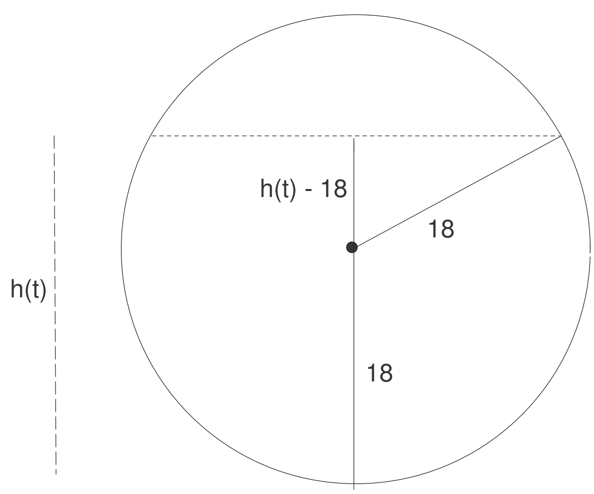

Figure 1.2: Problem 30, Section 1.1.

with $h(0) = 36$. Here $h(t)$ is the height of the upper surface of the fluid above the bottom of the sphere. This equation simplifies to

$$(36\sqrt{h} - h^{3/2})\, dh = -0.4\, dt,$$

a separated equation with general solution $h\sqrt{h}(60 - h) = -t + k$. Then $t = 0$ when $h = 36$ gives us $k = 5184$. The tank runs empty when $h = 0$, so $t = 5184$ seconds, about 86.4 minutes. This is the time it takes to drain this spherical tank.

31. (a) Let $r(t)$ be the radius of the exposed water surface and $h(t)$ the depth of the draining water at time t. Since cross sections of the cone are similar,

$$\pi r^2 \frac{dh}{dt} = -kA\sqrt{2gh},$$

with $h(0) = 9$. From similar triangles (Figure 1.3), $r/h = 4/9$, so $r = (4/9)h$. Substitute $k = 0.6$, $g = 32$ and $A = \pi(1/12)^2$ and simplify the resulting equation to obtain

$$h^{3/2} \frac{dh}{dt} = -27/160,$$

with $h(0) = 9$. This separable equation has the general solution given implicitly by

$$h^{5/2} = -\frac{27}{64}t + k.$$

Since $h(0) = 9$, then $k = 243$ and the tank empties out when $h = 0$, so

$$t = 243\left(\frac{64}{27}\right) = 576$$

seconds, about 9 minutes, 36 seconds.

(b) This problem is modeled like part (a), except now the cone is inverted. This changes the similar triangle proportionality (Figure 1.4) to

$$\frac{r}{9 - h} = \frac{4}{9}.$$

Then $r = (4/9)(9 - h)$. The separable differential equation becomes

$$\frac{(9 - h)^2}{\sqrt{h}}\, dh = -\frac{27}{160},$$

with $h(0) = 9$. This initial value problem has the solution

$$162\sqrt{h} - 12h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{27}{160}t + \frac{1296}{5}.$$

The tank runs dry at $h = 0$, which occurs when

$$t = \frac{160}{27}\left(\frac{1296}{5}\right) = 1536$$

seconds, about 25 minutes, 36 seconds.

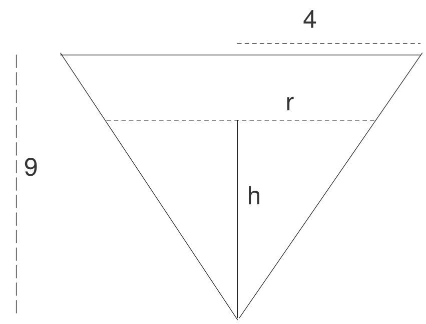

Figure 1.3: Problem 31(a), Section 1.1.

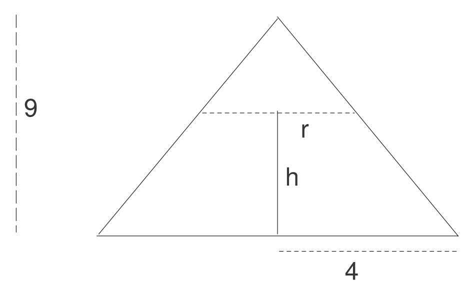

Figure 1.4: Problem 31(b), Section 1.1.

32. From the geometry of the cone and Torricelli's law,

$$\frac{dV}{dt} = \pi\left(\frac{16}{81}\right)h^2\frac{dh}{dt} = -\frac{(0.6)(8\pi)}{144}\sqrt{h-2}$$

when the drain hole is two feet above the vertex. With the drain hole at the bottom of the tank we get

$$\frac{dV}{dt} = \pi\left(\frac{16}{81}\right)^2 h^2\frac{dh}{dt} = -\frac{(0.6)(8\pi)}{144}\sqrt{h}.$$

If we know the rates of change of depth of the water in these two instances, then we can locate the drain hole height above the bottom of the tank, knowing the hole size, since

$$\pi\left(\frac{16}{81}\right)h^2\left(\frac{dh}{dt}\right)_1 = -kA\sqrt{2g(h-h_0)}$$

divided by

$$\pi\left(\frac{16}{81}\right)^2 h^2\left(\frac{dh}{dt}\right)_2 = -kA\sqrt{2gh}$$

yields

$$\frac{h-h_0}{\sqrt{h}} = \frac{(dh/dt)_1}{(dh/dt)_2} = r,$$

a known constant. We can therefore solve for h_0, the location of the hole above the bottom of the tank.

33. Begin with the logistic equation

$$P'(t) = aP(t) - bP(t)^2$$

in which a and b are positive constants. Then

$$\frac{dP}{dt} = (a-bP)P.$$

This is separable and we can write

$$\frac{1}{(a-bP)P}\,dP = dt.$$

Use a partial fractions decomposition to write

$$\left(\frac{1}{a}\frac{1}{P} + \frac{b}{a}\frac{1}{a-bP}\right)dP = dt.$$

Integrate to obtain

$$\frac{1}{a}\ln(P) - \frac{1}{a}\ln(a-bP) = t + c.$$

Here we assume that $P(t) > 0$ and $a - bP(t) > 0$. Write this equation as

$$\ln\left(\frac{P}{a - bP}\right) = at + k,$$

with $k = ac$ still a constant to be determined. Then

$$\frac{P}{a - bP} = e^{at+k} = e^k e^{at} = Ke^{at},$$

where $K = e^k$ is the constant to be determined. Now $P(0) = p_0$, so

$$K = \frac{p_0}{a - bp_0}.$$

Then

$$\frac{P}{a - bP} = \frac{p_0}{a - bp_0} e^{at}.$$

It is a straightforward algebraic manipulation to solve for P and obtain

$$P(t) = \frac{ap_0}{a - bp_0 + bp_0 e^{at}} e^{at}.$$

Notice that $P(t)$ is a strictly increasing function. Further, by multiplying numerator and denominator by e^{-at}, and using the fact that $a > 0$, we have

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{ap_0}{(a - bp_0)e^{-at} + bp_0} \\ &= \frac{ap_0}{bp_0} = \frac{a}{b}. \end{aligned}$$

34. With a and b taking on the given values, and $p_0 = 3,929,214$, the population in 1790, we obtain the logistic model for the United States population growth:

$$P(t) = \frac{123,141.5668}{0.03071576577 + 0.0006242342282 e^{0.03134t}} e^{0.03134t}.$$

Table 1.1 shows compares the population figures given by $P(t)$ with the actual numbers, together with the percent error (positive if $P(t)$ exceeds the actual population, negative if $P(t)$ is an underestimate).

An exponential model can also be constructed as $Q(t) = Ae^{kt}$. Then

$$A = Q(0) = 3,929,214,$$

the initial (1790) population. To find k, use the fact

$$Q(10) = 5308483 = 3929214 e^{10k}$$

year	population	$P(t)$	percent error	$Q(t)$	percent error
1790	3,929,214	3,929,214	0	3,929,214	0
1800	5,308,483	5,336,313	0.52	5,308,483	0
1810	7,239,881	7,228,471	-0.16	7,179,158	-0.94
1820	9,638,453	9,757,448	1.23	9,689,468	0.53
1830	12,886,020	13,110,174	1.90	13,090,754	1.75
1840	17,069,453	17,507,365	2.57	17,685,992	3.61
1850	23,191,876	23,193,639	0.008	23,894,292	3.03
1860	31,443,321	30,414,301	-3.27	32,281,888	2.67
1870	38,558,371	39,374,437	2.12	43,613,774	13.11
1880	50,189,209	50,180,383	-0.018	58,923,484	17.40
1890	62,979,766	62,772,907	-0.33	79,073,491	26.40
1900	76,212,168	76,873,907	0.87	107,551,857	41.12
1910	92,228,496	91,976,297	-0.27	145,303,703	57.55
1920	106,021,537	107,398,941	1.30	196,312,254	85.16
1930	123,202,624	122,401,360	-0.65		
1940	132,164,569	136,320,577	3.15		
1950	151,325,798	148,679,224	-1.75		
1960	179,323,175	159,231,097	-11.2		
1970	203,302,031	167,943,428	-17.39		
1980	226,547,042	174,940,040	-22.78		

Table 1.1: Census and model data for Problems 33 and 34

to solve for k, obtaining

$$k = \frac{1}{10} \ln \left(\frac{5308483}{3929214} \right) \approx 0.03008667012.$$

Thus the exponential model determined using these two data points (1790 and 1800) is

$$Q(t) = 3929214e^{0.03008667012t}.$$

Population figures predicted by this model are also included in Table 1.1, along with percentage errors. Notice that the logistic model remains quite accurate until 1960, at which time the error increases dramatically for the next three years. The exponential model becomes increasingly inaccurate by 1870, after which the error rapidly becomes so large that it is not worth computing further. Exponential models do not work well over time with complex populations, such as fish in the ocean or countries throughout the world.

1.2 Linear Equations

1. With $p(x) = -3/x$, an integrating factor is
$$e^{\int p(x)\,dx} = e^{-3\ln(x)} = x^{-3}.$$
Multiply the differential equation by x^{-3} to obtain
$$\frac{d}{dx}(yx^{-3}) = \frac{2}{x}.$$
A routine integration gives us $yx^{-3} = 2\ln(x) + c$, or
$$y = cx^3 + 2x^3 \ln|x|$$
for $x \neq 0$.

2. $e^{\int dx} = e^x$ is an integrating factor. Multiply the differential equation by e^x to obtain
$$y'e^x + ye^x = (ye^x)' = \frac{1}{2}\left(e^{2x} - 1\right).$$
Integrate to obtain
$$ye^x = \frac{1}{4}e^{2x} - \frac{1}{2}x + c.$$
Then
$$y = \frac{1}{4}e^x - \frac{1}{2}xe^{-x} + ce^{-x}.$$

3. $e^{\int 2\,dx} = e^{2x}$ is an integrating factor. Multiply the differential equation by e^{2x} to obtain
$$y'e^{2x} + 2y = (ye^{2x})' = xe^{2x}.$$
Integrate to obtain
$$ye^{2x} = \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c.$$
The general solution is
$$y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}.$$

4. An integrating factor is
$$e^{\int \sec(x)\,dx} = e^{\ln|\sec(x)+\tan(x)|} = \sec(x) + \tan(x).$$
Multiply the differential equation by $\sec(x) + \tan(x)$ to obtain
$$\begin{aligned} &y'(\sec(x) + \tan(x)) + (\sec(x)\tan(x) + \sec^2(x))y \\ &= (y(\sec(x) + \tan(x)))' = \cos(x)(\sec(x) + \tan(x)) \\ &= 1 + \sin(x). \end{aligned}$$

Integrate this equation to obtain

$$y(\sec(x) + \tan(x)) = x - \cos(x) + k.$$

Multiply both sides of this equation by

$$\frac{1}{\sec(x) + \tan(x)} = \frac{\cos(x)}{1 + \sin(x)}$$

to obtain

$$\begin{aligned} y &= (x - \cos(x) + k)\left(\frac{\cos(x)}{1 + \sin(x)}\right) \\ &= \frac{x\cos(x) - \cos^2(x) + k\cos(x)}{1 + \sin(x)}. \end{aligned}$$

5. An integrating factor is $e^{\int -2\,dx} = e^{-2x}$. Multiply the differential equation by e^{-2x} to obtain

$$y'e^{-2x} - 2ye^{-2x} = (ye^{-2x})' = -8x^2e^{-2x}.$$

Integrate to obtain

$$ye^{-2x} = \int -8x^2e^{-2x}\,dx = 4x^2e^{-2x} + 4xe^{-2x} + 2e^{-2x} + c.$$

The general solution is

$$y = 4x^2 + 4x + 2 + ce^{2x}.$$

6. $e^{\int 3\,dx} = e^{3x}$ is an integrating factor. Multiply the differential equation by e^{3x} to obtain

$$y'e^{3x} + 3ye^{3x} = (ye^{3x})' = 5e^{5x} - 6e^{3x}.$$

Integrate to obtain the general solution

$$ye^{3x} = e^{5x} - 2e^{3x} + c.$$

The general solution is

$$y = e^{2x} - 2 + ce^{-3x}.$$

Now we need

$$y(0) = 1 - 2 + c = 2,$$

so $c = 3$. The initial value problem has the solution

$$y = e^{2x} - 2 + 3e^{-3x}.$$

7. Notice that, if we multiply the differential equation by $x-2$, we obtain

$$y'(x-2)+y=((x-2)y)'=3x(x-2).$$

Integrate to obtain

$$(x-2)y=x^3-3x^2+c.$$

The general solution is

$$y=\frac{1}{x-2}(x^3-3x^2+c).$$

Now

$$y(3)=27-27+c=4$$

so the initial value problem has the solution

$$y=\frac{x^3-3x^2+4}{x-2}=x^2-x-2.$$

8. Multiply the differential equation by the integrating factor e^{-x} to obtain

$$(ye^{-x})'=2e^{3x}.$$

Integrate to obtain

$$ye^{-x}=\frac{2}{3}e^{3x}+c.$$

The general solution is

$$y=\frac{2}{3}e^{4x}+ce^{x}.$$

Then

$$y(0)=-3=\frac{2}{3}+c$$

so $c=-11/3$ and the initial value problem has the solution

$$y=\frac{2}{3}e^{4x}-\frac{11}{3}e^{x}$$

9. An integrating factor is

$$e^{\int(2/(x+1))\,dx}=e^{2\ln|x+1|}=e^{\ln((x+1)^2)}=(x+1)^2.$$

Multiply the differential equation by $(x+1)^2$ to obtain

$$(x+1)^2y'+2(x+1)y=((x+1)^2y)'=3(x+1)^2.$$

Integrate to obtain

$$(x+1)^2y=(x+1)^3+c.$$

Then

$$y=(x+1)+\frac{c}{(x+1)^2}.$$

Now

$$y(0) = 5 = 1 + c$$

so $c = 4$ and the solution of the initial value problem is

$$y = x + 1 + \frac{4}{(x+1)^2}.$$

10. An integrating factor is

$$e^{\int (5/9x)\,dx} = e^{(5/9)\ln(x)} = e^{\ln(x^{5/9})} = x^{5/9}.$$

Multiply the differential equation by $x^{5/9}$ to obtain

$$(yx^{5/9})' = 3x^{32/9} + x^{14/9}.$$

Integrate to obtain

$$yx^{5/9} = \frac{27}{41}x^{41/9} + \frac{9}{23}x^{23/9} + c.$$

Then

$$y = \frac{27}{41}x^4 + \frac{9}{23}x^2 + cx^{-5/9}.$$

We need

$$y(-1) = 4 = \frac{27}{41} + \frac{9}{23} - c,$$

so $c = -2782/943$. The solution is

$$y = \frac{27}{41}x^4 + \frac{9}{23}x^2 - \frac{2782}{943}x^{-5/9}$$

11. Let (x, y) be a point on the curve. The tangent line at (x, y) must pass through $(0, 2x^2)$, hence must have slope $(y - 2x^2)/x$. But this slope is y', so we have the differential equation

$$y' = \frac{y - 2x^2}{x}.$$

This is the linear differential equation

$$y' - \frac{1}{x}y = -2x,$$

which has the general solution $y = -2x^2 + cx$.

12. If $A(t)$ is the amount of salt in the tank at time $t \geq 0$, then

$$\begin{aligned}\frac{dA}{dt} &= \text{ rate salt is added } - \text{ rate salt is removed}\\ &= 6 - 2\left(\frac{A(t)}{50+t}\right),\end{aligned}$$

and the initial condition is $A(0) = 28$.

This differential equation is linear:

$$A' + \frac{2}{50+t}A = 6,$$

with integrating factor $(50+t)^2$. The general solution is

$$A(t) = 2(50+t) + \frac{C}{(50+t)^2},$$

The initial condition gives us $C = -180,000$, so

$$A(t) = 2(50+t) - \frac{180000}{(50+t)^2}.$$

The tank contains 100 gallons when $t = 50$ and $A(50) = 176$ pounds of salt.

13. If $A_1(t)$ and $A_2(t)$ are the amounts of salt in tanks one and two, respectively, at time t, we have

$$A_1'(t) = \frac{5}{2} - \frac{5A_1(t)}{100}; A_1(0) = 20$$

and

$$A_2'(t) = \frac{5A_1(t)}{100} - \frac{5A_2(t)}{150}; A_2(0) = 90.$$

Solve the first initial value problem to obtain

$$A_1(t) = 50 - 30e^{-t/20}.$$

Substitute this into the problem for $A_2(t)$ to obtain

$$A_2' + \frac{1}{30}A_2 = \frac{5}{2} - \frac{3}{2}e^{-t/20}; A_2(0) = 90.$$

Solve this to obtain

$$A_2(t) = 75 + 90e^{-t/20} - 75e^{-t/30}.$$

Tank 2 has its minimum when $A_2'(t) = 0$, hence when

$$2.5e^{-t/30} - 4.5e^{-t/20} = 0.$$

Then $e^{t/60} = 9/5$, or $t = 60\ln(9/5)$. Then

$$A_2(t)_{\text{min}} = A_2(60\ln(9/5)) = \frac{5450}{81}$$

pounds.

1.3 Exact Equations

In the following we assume that the differential equation has the form $M(x, y) + N(x, y)y' = 0$, or, in differential form, $M\,dx + N\,dy = 0$.

1. Since
$$\frac{\partial M}{\partial y} = 4y + e^{xy} + xye^{xy} = \frac{\partial N}{\partial x}$$
for all x and y, the equation is exact in the entire plane. One way to find a potential function is to integrate
$$\frac{\partial \varphi}{\partial x} = M(x, y) = 2y^2 + ye^{xy}$$
with respect to x to obtain
$$\varphi(x, y) = 2xy^2 + e^{xy} + \alpha(y).$$
Then we need
$$\frac{\partial \varphi}{\partial y} = 4xy + xe^{xy} + \alpha'(y) = N(x, y) = 4xy + xe^{xy} + 2y.$$
This requires that $\alpha'(y) = 2y$ so we may choose $\alpha(y) = y^2$. A potential function has the form
$$\varphi(x, y) = 2xy^2 + e^{xy} + y^2.$$
The general solution is implicitly defined by
$$\varphi(x, y) = 2xy^2 + e^{xy} + y^2 = c.$$
We could have also started by integrating $\partial N/\partial y = 4xy + xe^{xy} + 2y$ with respect to y.

2. Since $\partial M/\partial y = 4x = \partial N/\partial x$ for all x and y, the equation is exact in the plane. We can find a potential function by integrating
$$\frac{\partial \varphi}{\partial y} = 2x^2 + 3y^2$$
with respect to y to obtain
$$\varphi(x, y) = 2x^2y + y^3 + \beta(x).$$
Then
$$\frac{\partial \varphi}{\partial x} = 4xy + \beta'(x) = 4xy + 2x,$$
so $\beta'(x) = 2x$ and we can choose $\beta(x) = x^2$. A potential function is
$$\varphi(x, y) = 2x^2y + y^3 + x^2$$
and the general solution is defined implicitly by
$$2x^2y + y^3 + x^2 = c.$$

3. $\partial M/\partial y = 4 + 2x^2$ and $\partial N/\partial x = 4x$, so this equation is not exact.

4.
$$\frac{\partial M}{\partial y} = -2\sin(x+y) - 2x\cos(x+y) = \frac{\partial N}{\partial x}$$
so the equation is exact over the plane. Routine integrations yield the potential function is $\varphi(x, y) = 2x\cos(x+y)$ and the general solution is implicitly defined by $2x\cos(x+y) = c$.

5. $\partial M/\partial y = 1 = \partial N/\partial x$, so the equation is exact for all (x, y) with $x \neq 0$, where the equation is not defined. Integrate $\partial\varphi/\partial x = M$ or $\partial\varphi/\partial y = N$ to obtain the potential function
$$\varphi(x, y) = \ln|x| + xy + y^3.$$
The general solution is defined implicitly by
$$\varphi(x, y) = \ln|x| + xy + y^3 = c$$
for $x \neq 0$.

6. For the equation to be exact, we need
$$\frac{\partial M}{\partial y} = \alpha x y^{\alpha-1} = \frac{\partial N}{\partial x} = -2xy^{\alpha-1}.$$
This holds if $\alpha = -2$. By integrating, we find the potential function $\varphi(x, y) = x^3 + x^2/2y^2$, so the general solution is defined implicitly by
$$x^3 + \frac{x^2}{2y^2} = c.$$

7. For exactness we need
$$\frac{\partial M}{\partial y} = 6xy^2 - 3 = \frac{\partial N}{\partial x} = -3 - 2\alpha xy^2$$
and this requires that $\alpha = -3$. By integration, we find a potential function $\varphi(x, y) = x^2y^3 - 3xy - 3y^2$. The general solution is implicitly defined by
$$x^2y^3 - 3xy - 3y^2 = c.$$

8. Compute
$$\frac{\partial M}{\partial y} = 2 - 2y\sec^2(xy^2) - 2xy^3\sec^2(xy^2)\tan(xy^2)$$
and
$$\frac{\partial N}{\partial x} = 2 - 2y\sec^2(xy^2) - 2xy^3\sec^2(xy^2)\tan(xy^2).$$

Since these partial derivatives are equal for all x and y for which the functions are defined, the differential equation is exact for such x and y. To find a potential function, we can start by integrating $\partial\varphi/\partial x = 2y - y^2\sec^2(xy^2)$ with respect to x to obtain

$$\varphi(x,y) = 2xy - \tan(xy^2) + \alpha(y).$$

Now we need

$$\begin{aligned}\frac{\partial\varphi}{\partial y} &= 2x - 2xy\sec^2(xy^2)\\ &= 2x - 2xy\sec^2(xy^2) + \alpha'(y).\end{aligned}$$

This requires that $\alpha'(y) = 0$ and we may choose $\alpha(y) = 0$. A potential function is

$$\varphi(x,y) = 2xy - \tan(xy^2).$$

The general solution is implicitly defined by

$$2xy - \tan(xy^2) = c.$$

For the initial condition we need $y = 2$ when $x = 1$, which requires that

$$2(2) - \tan(4) = c.$$

The unique solution of the initial value problem is implicitly defined by

$$2xy - \tan(xy^2) = 4 - \tan(4).$$

9. Since $\partial M/\partial y = 12y^3 = \partial N \partial x$, the differential equation is exact for all x and y. Straightforward integrations yield the potential function

$$\varphi(x,y) = 3xy^4 - x.$$

The general solution is implicitly defined by

$$3xy^4 - x = c.$$

For the initial condition, we need $y = 2$ when $x = 1$, so

$$3(1)(2^4) - 1 = 47 = c.$$

The initial value problem has the unique solution implicitly defined by

$$3xy^4 - x = 47.$$

10. Compute

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{1}{x}e^{y/x} - \frac{1}{x}e^{y/x} - \frac{y}{x^2}e^{y/x}\\ &= -\frac{y}{x^2}e^{y/x} = \frac{\partial N}{\partial x},\end{aligned}$$

so the differential equation is exact for all $x \neq 0$ and all y. For a potential function, begin with

$$\frac{\partial \varphi}{\partial y} = e^{y/x}$$

and integrate with respect to y to obtain

$$\varphi(x, y) = xe^{y/x} + \beta(x).$$

Then

$$\frac{\partial \varphi}{\partial x} = 1 + e^{y/x} - \frac{y}{x}e^{y/x} = e^{y/x} - \frac{y}{x}e^{y/x} + \beta'(x).$$

This requires that $\beta'(x) = 1$ so choose $\beta(x) = x$. Then

$$\varphi(x, y) = xe^{y/x} + x.$$

The general solution is implicitly defined by

$$xe^{y/x} + x = c.$$

For the initial value problem, we need to choose c so that

$$e^{-5} + 1 = c.$$

The solution of the initial value problem is implicitly defined by

$$xe^{y/x} + x = 1 + e^{-5}.$$

11. Compute

$$\frac{\partial M}{\partial y} = -2x\sin(2y - x) - 2\cos(2y - x) = \frac{\partial N}{\partial x},$$

so the differential equation is exactly. For a potential function, integrate

$$\frac{\partial \varphi}{\partial y} = -2x\cos(2y - x)$$

with respect to y to get

$$\varphi(x, y) = -x\sin(2y - x) + \alpha(x).$$

Then we must have

$$\begin{aligned}\frac{\partial \varphi}{\partial x} &= x\cos(2y - x) - \sin(2y - x) \\ &= -\sin(2y - x) + x\cos(2y - x) + \alpha'(x).\end{aligned}$$

Then $\alpha'(x) = 0$ and we may choose $\alpha(x) = 0$ to obtain

$$\varphi(x, y) = -x\sin(2y - x).$$

The general solution has the form

$$-x\sin(2y-x)=c.$$

For $y(\pi/12)=\pi/8$, we need

$$-\frac{\pi}{12}\sin\left(\frac{\pi}{4}-\frac{\pi}{12}\right)=-\frac{\pi}{12}\sin(\pi/6)=-\frac{\pi}{24}=c.$$

The solution of the initial value problem is implicitly defined by

$$x\sin(2y-x)=\frac{\pi}{24}.$$

12. The equation is exact over the entire plane because

$$\frac{\partial M}{\partial y}=e^y=\frac{\partial N}{\partial x}.$$

Integrate

$$\frac{\partial\varphi}{\partial x}=e^y$$

with respect to x to get

$$\varphi(x,y)=xe^y+\alpha(y).$$

Then we need

$$\frac{\partial\varphi}{\partial y}=xe^y+\alpha'(y)=xe^y-1.$$

Then $\alpha'(y)=-1$ and we can take $\alpha(y)=-y$. Then

$$\varphi(x,y)=xe^y-y.$$

The general solution is implicitly defined by

$$xe^y-y=c.$$

For the initial condition, we need $y=0$ when $x=5$, so choose $c=5$ to obtain the implicitly defined solution

$$xe^y-y=5.$$

13. $\varphi+c$ is also a potential function if φ is because

$$\frac{\partial(\varphi+c)}{\partial x}=\frac{\partial\varphi}{\partial x}$$

and

$$\frac{\partial(\varphi+c)}{\partial y}=\frac{\partial\varphi}{\partial y}$$

Any function defined implicitly by $\varphi(x,y)=k$ is also defined by $\varphi(x,y)+c=k$, because, if k can assume any real value, so can $k-c$ for any c.

14. (a)

$$\frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = -1$$

so this differential equation is not exact over any rectangle in the plane.

(b) Multiply the differential equation by x^{-2} to obtain

$$yx^{-2} - x^{-1}y' = 0.$$

This is exact over any rectangle not containing $x = 0$, because

$$\frac{\partial M^*}{\partial y} = x^{-2} = \frac{\partial N^*}{\partial x}.$$

This equation has potential function $\varphi(x, y) = -yx^{-1}$, so the general solution is defined implicitly by

$$-yx^{-1} = c.$$

(c) If we multiply the differential equation by y^{-2} we obtain

$$y^{-1} - xy^{-2}y' = 0.$$

This is exact on any region not containing $y = 0$ because

$$\frac{\partial M^{**}}{\partial y} = -y^{-2} = \frac{\partial N^{**}}{\partial x}.$$

This has potential function $\varphi(x, y) = xy^{-1}$, so the differential equation has the general solution

$$xy^{-1} = c.$$

(d) Multiply the differential equation by xy^{-2} to obtain

$$xy^{-2} - x^2y^{-3}y' = 0.$$

Now

$$\frac{\partial M^{***}}{\partial y} = -2xy^{-3} = \frac{\partial N^{***}}{\partial x}$$

so this differential equation is exact. Integrate $\partial\varphi/\partial x = xy^{-2}$ with respect to x to obtain

$$\varphi(x, y) = \frac{1}{2}x^2y^{-2} + \beta(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = -x^2y^{-3} + \beta'(y) = -x^2y^{-3}$$

so choose $\beta(y) = 0$. The general solution in this case is given implicitly by

$$x^2y^{-2} = c.$$

(e) As a linear equation, we have

$$y' - \frac{1}{x}y = 0,$$

or $xy' - y = (x^{-1}y)' = 0$. This has general solution defined implicitly by $x^{-1}y = c$.

(f) The general solutions obtained in (b) through (e) are the same. For example, in (b) we obtained $-yx^{-1} = c$. Since c is an arbitrary constant, this can be written $y = kx$. In (d) we obtained $x^2y^{-2} = c$. This can be written $y^2 = Cx^2$, or $y = kx$.

15. Multiply the differential equation by $\mu(x, y) = x^a y^b$ to obtain

$$x^{a+1}y^{b+1} + x^a y^{b-3/2} + x^{a+2}y^b y' = 0.$$

For this to be exact, we need

$$\begin{aligned}\frac{\partial M}{\partial y} &= (b+1)x^{a+1}y^b + \left(b - \frac{3}{2}\right)x^a y^{b-5/2} \\ &= \frac{\partial N}{\partial x} = (a+2)x^{a+1}y^b.\end{aligned}$$

Divide this by $x^a y^b$ to require that

$$(b+1)x + \left(b - \frac{3}{2}\right)y^{-5/2} = (a+2)x.$$

This will be true for all x and y if we let $b = 3/2$, and then choose a so that $(b+1)x = (a+2)x$, so $b+1 = a+2$. Therefore

$$a = \frac{1}{2} \text{ and } b = \frac{3}{2}.$$

Multiply the original differential equation by $\mu(x, y) = x^{1/2}y^{3/2}$ to obtain

$$x^{3/2}y^{5/2} + x^{1/2} + x^{5/2}y^{3/2}y' = 0.$$

Integrate $\partial\varphi/\partial y = x^{5/2}y^{3/2}$ to obtain

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + \beta(x).$$

Then we need

$$\frac{\partial \varphi}{\partial x} = x^{3/2}y^{5/2} + \beta'(x) = x^{3/2}y^{5/2} + x^{1/2}.$$

Then $\beta(x) = 2x^{3/2}/3$ and a potential function is

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2}.$$

The general solution of the original differential equation is

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2} = c.$$

The differential equation multiplied by the integrating factor has the same solutions as the original differential equation because the integrating factor is assumed to be nonzero. Thus we must exclude $x = 0$ and $y = 0$, where $\mu = 0$.

16. Multiply the differential equation by $x^a y^b$:

$$2x^a y^{b+2} - 9x^{a+1}y^{b+1} + (3x^{a+1}y^{b+1} - 6x^{a+2}y^b)y' = 0.$$

For this to be exact, we must have

$$\begin{aligned}\frac{\partial M}{\partial y} &= (b+2)2x^a y^{b+1} - 9(b+1)x^{a+1}y^b \\ &= \frac{\partial N}{\partial x} = 3(a+1)x^a y^{b+1} - 6(a+2)x^{a+1}y^b.\end{aligned}$$

Divide by $x^a y^b$ to obtain, after some rearrangement,

$$(2(b+2) - 3(a+1))y = ((9(b+1) - 6(a+2))x.$$

Since x and y are independent, this equation can hold only if the coefficients of x and y are zero, giving us two equations for a and b:

$$-3a + 2b = -1, -6a + 9b = 3.$$

Then $a = b = 1$, so $\mu(x, y) = xy$ is an integrating factor. Multiply the differential equation by xy:

$$2xy^3 - 9x^2y^2 + (3x^2y^2 - 6x^3y)y' = 0.$$

It is routine to check that this equation is exact. For a potential function, integrate

$$\frac{\partial \varphi}{\partial x} = 2xy^3 - 9x^2y^2$$

with respect to x to get

$$\varphi(x, y) = x^2y^3 - 3x^3y^2 + \beta(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = 3x^2y^2 - 6x^3y + \beta'(y).$$

We may choose $\beta(y) = 0$, so $\varphi(x, y) = x^2y^3 - 3x^3y^2$. The general solution is implicitly defined by

$$x^2y^3 - 3x^3y^2 = c.$$

1.4 Homogeneous, Bernoulli and Riccati Equations

1. This is a Riccati equation with solution $S(x) = x$ (by inspection). Put $y = x + 1/z$ and substitute to obtain

$$2 - \frac{z'}{z^2} = \frac{1}{x^2}\left(x + \frac{1}{z}\right)^2 - \frac{1}{x}\left(x + \frac{1}{z}\right) + 1.$$

Simplify this to obtain

$$z' + \frac{1}{x}z = -\frac{1}{x^2}.$$

This linear differential equation can be written $(xz)' = -1/x$ and has the solution

$$z = -\frac{\ln(x)}{x} + \frac{c}{x}.$$

Then

$$y = x + \frac{x}{c - \ln(x)}$$

for $x > 0$.

2. This is a Bernoulli equation with $\alpha = -4/3$. Put $v = y^{7/3}$, or $y = v^{3/7}$. Substitute this into the differential equation to get

$$\frac{3}{7}v^{-4/7}v' + \frac{1}{x}v^{3/7} = \frac{2}{x^3}v^{-4/7}.$$

This simplifies to the linear equation

$$v' + \frac{7}{3x}v = \frac{14}{3x^2}.$$

This has integrating factor $x^{7/3}$ and can be written

$$(vx^{7/3})' = \frac{14}{3}x^{1/3}.$$

Integration yields

$$vx^{7/3} = \frac{7}{2}x^{4/3} + c.$$

Since $v = y^{7/3}$, we obtain

$$2y^{7/3}x^{7/3} - 7x^{4/3} = k.$$

This implicitly defined the general solution.

3. This is a Bernoulli equation with $\alpha = 2$ and we obtain the general solution

$$y = \frac{1}{1 + ce^{x^2/2}}.$$

4. This equation is homogeneous. With $y = xu$, we obtain

$$u + xu' = u + \frac{1}{u}.$$

Then

$$x\frac{du}{dx} = \frac{1}{u},$$

a separable equation. Write

$$u\,du = \frac{1}{x}\,dx.$$

Integrate to obtain

$$u^2 = 2\ln|x| + c.$$

Then

$$\frac{y^2}{x^2} = 2\ln|x| + c$$

implicitly defines the general solution of the original differential equation.

5. This differential equation is homogeneous, and $y = xu$ yields the general solution implicitly defined by

$$y\ln|y| - x = cy.$$

6. The differential equation is Riccati and we see one solution $S(x) = 4$. We obtain the general solution

$$y = 4 + \frac{6x^3}{c - x^3}.$$

7. This equation is exact, with general solution defined by

$$xy - x^2 - y^2 = c.$$

8. The differential equation is homogeneous, and $y = xu$ yields the general solution defined by

$$\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = cx.$$

9. The differential equation is Bernoulli, with $\alpha = -3/4$. The general solution is given by

$$5x^{7/4}y^{7/4} + 7x^{-5/4} = c.$$

10. The differential equation is homogeneous and $y = xu$ yields

$$\frac{2\sqrt{3}}{\sqrt{3}}\arctan\left(\frac{2y - x}{\sqrt{3}x}\right) = \ln|x| + c.$$

11. The equation is Bernoulli with $\alpha = 2$. We obtain
$$y = 2 + \frac{2}{cx^2 - 1}.$$

12. The equation is homogeneous and $y = xu$ yields
$$\frac{1}{2}\frac{x^2}{y^2} = \ln|x| + c.$$

13. The equation is Riccati with one solution $S(x) = e^x$. The general solution is
$$y = \frac{2e^x}{ce^{2x} - 1}.$$

14. The equation is Bernoulli with $\alpha = 2$ and general solution
$$y = \frac{2}{3 + cx^2}.$$

15. For the first part,
$$F\left(\frac{ax + by + c}{dx + py + r}\right) = F\left(\frac{a + b(y/x) + c/x}{d + p(y/x) + r/x}\right) = f\left(\frac{y}{x}\right)$$
if and only if $c = r = 0$.

    Now suppose $x = X + h$ and $y = Y + k$. Then
$$\frac{dY}{dX} = \frac{dY}{dx}\frac{dx}{dX} = \frac{dy}{dx}$$
so
$$\begin{aligned}\frac{dY}{dX} &= F\left(\frac{a(X + h) + b(Y + k) + c}{d(X + h) + p(Y + k) + r}\right)\\ &= F\left(\frac{aX + bY + c + ah + bk + c}{dX + pY + r + dh + pk + r}\right)\end{aligned}$$
This equation is homogeneous exactly when
$$ah + bk = -c \text{ and } dh + pk = -r.$$
This two by two system has a solution when the determinant of the coefficients is nonzero: $ap - bd \neq 0$.

16. Here $a = 0, b = 1, c = -3$ and $d = p = 1, r = -1$. Solve
$$k = 3, h + k = 1$$
to obtain $k = 3$ and $h = -2$. Thus let $x = X - 2, y = Y + 3$ to obtain
$$\frac{dY}{dX} = \frac{Y}{X + Y},$$

a homogeneous equation. Letting $U = Y/X$ we obtain, after some manipulation,

$$\frac{1+U}{U}\,dU = \frac{1}{X}\,dX,$$

a separable equation with general solution

$$U\ln|U| - 1 = -U\ln|X| + KU,$$

in which K is the arbitrary constant. In terms of x and y,

$$(y-3)\ln|y-3| - (x+2) = K(y-3).$$

17. Set $x = X+2, y = Y-3$ to obtain

$$\frac{dY}{dX} = \frac{3X-Y}{X+Y}.$$

This homogeneous equation has general solution (in terms of x and y)

$$3(x-2)^2 - 2(x-2)(y+3) - (y+3)^2 = K.$$

18. With $x = X-5$ and $y = Y-1$ we obtain

$$(x+5)^2 + 4(x+5)(y+1) - (y+1)^2 = K.$$

19. with $x = X+2$ and $y = Y-1$ we obtain

$$(2x+y-3)^2 = K(y-x+3).$$

1.5 Additional Applications

1. Once released, the only force acting on the ballast bag is due to gravity. If $y(t)$ is the distance from the bag to the ground at time t, then $y'' = -g = -32$, with $y(0) = 4$. With two integrations, we obtain

$$y'(t) = 4 - 32t \text{ and } y(t) = 342 + 4t - 16t^2.$$

The maximum height is reached when $y'(t) = 0$, or $t = 1/8$ second. This maximum height is $y(1/8) = 342.25$ feet. The bag remains aloft until $y(t) = 0$, or $-16t^2 + 4t + 342 = 0$. This occurs at $t = 19/4$ seconds, and the bag hits the ground with speed $|y'(19/4)| = 148$ feet per second.

2. With a gradient of $7/24$ the plane is inclined at an angle θ for which $\sin(\theta) = 7/25$ and $\cos(\theta) = 24/25$. The velocity of the box satisfies

$$\frac{48}{32}\frac{dv}{dt} = -48\left(\frac{24}{25}\right)\left(\frac{1}{3}\right) + 48\left(\frac{7}{25}\right) - \frac{3}{2}v;\, v(0) = 16.$$

Solve this initial value problem to obtain

$$v(t) = \frac{432}{25}e^{-t} - \frac{32}{25}$$

feet per second. This velocity reaches zero when $t_s = \ln(27/2)$ seconds. The box will travel a distance of

$$\begin{aligned} s(t_s) &= \int_0^{t_s} v(\xi)\,d\xi = \frac{432}{25}(1 - e^{-t_s}) - \frac{32}{25}t_s \\ &= \frac{432}{25}\left(1 - \frac{2}{27}\right) - \frac{32}{25}\ln\left(\frac{27}{2}\right) \approx 12.7 \end{aligned}$$

feet.

3. Until the parachute is opened at $t = 4$ seconds, the velocity $v(t)$ satisfies the initial value problem

$$\left(\frac{192}{32}\right)\frac{dv}{dt} = 192 - 6v; v(0) = 0.$$

This has solution $v(t) = 32(1 - e^{-t})$ for $0 \le t \le 4$. When the parachute opens at $t = 4$, the skydiver has a velocity of $v(4) = 32(1 - e^{-4})$ feet per second. Velocity with the open parachute satisfies the initial value problem

$$\left(\frac{192}{32}\right)\frac{dv}{dt} = 192 - 3v^2, v(4) = 32(1 - e^{-4}) \text{ for } t \ge 4.$$

This differential equation is separable and can be integrated using partial fractions:

$$\int \left[\frac{1}{v+8} - \frac{1}{v-8}\right] dv = -\int 8t\,dt.$$

This yields

$$\ln\left(\frac{v+8}{v-8}\right) = -8t + \ln\left(\frac{5 - 4e^{-4}}{3 - 4e^{-4}}\right) + 32.$$

Solve for $v(t)$ to obtain

$$v(t) = \frac{8(1 + ke^{-8(t-4)})}{1 - ke^{-8(t-4)}} \text{ for } t \ge 4.$$

We find using the initial condition that

$$k = \frac{3 - 4e^{-4}}{5 - 4e^{-4}}.$$

Terminal velocity is $\lim_{t\to\infty} v(t) = 8$ feet per second. The distance fallen is

$$s(t) = \int_0^t v(\xi)\,d\xi = 32(t - 1 + e^{-t})$$

for $0 \le t \le 4$, while

$$s(t) = 32(3 + e^{-4}) + 8(t - 4) + 2\ln(1 - ke^{-8(t-4)}) - 2\ln\left(\frac{2}{5 - 4e^{-4}}\right)$$

for $t \ge 4$.

4. When fully submerged the buoyant force will be $F_B = (1)(2)(3)(62.5) = 375$ pounds upward. The mass is $m = 384/32 = 12$ slugs. The velocity $v(t)$ of the sinking box satisfies

$$12\frac{dv}{dt} = 384 - 375 - \frac{1}{2}v; v(0) = 0.$$

This linear problem has the solution

$$v(t) = 18(1 - e^{-t/24}).$$

In t seconds the box has sunk $s(t) = 18(t + 24e^{-t/24} - 24)$ feet. From $v(t)$ we find the terminal velocity

$$\lim_{t\to\infty} v(t) = 18$$

feet per second. To answer the question about velocity when the box reaches the bottom $s = 100$, we would normally solve $s(t) = 100$ and substitute this t into the velocity. This would require a numerical solution, which can be done. However, there is another approach we can also use. Find t_* so that $v(t_*) = 10$ feet per second, and calculate $s(t_*)$ to see how far the box has fallen. With this approach we solve $18(1 - e^{-t/24}) = 10$ to obtain $t_* = 24\ln(9/4)$ seconds. Now compute

$$s(t_*) = 432\ln(9/4) - 240 \approx 110.3$$

feet. Therefore at the bottom $s = 100$, the box has not yet reached a velocity of 10 feet per second.

5. If the box loses 32 pounds of material on impact with the bottom, then $m = 11$ slugs. Now

$$11\frac{dv}{dt} = -352 + 375 - \frac{1}{2}v; v(0) = 0$$

in which we have taken up as the positive direction. This gives us

$$v(t) = 46(1 - e^{-t/22})$$

so the distance traveled up from the bottom is

$$s(t) = 46(t + 22e^{-t/22} - 22)$$

feet. Solve $s(t) = 100$ numerically to obtain $t \approx 10.56$ seconds. The surfacing velocity is approximately $v(10.56) \approx 17.5$ feet per second.

6. The statement of gravitational attraction inside the Earth gives $v'(t) = -kr$, where r is the distance to the Earth's center. When $r = R$, the acceleration is g, so $k = -g/R$ and $v'(t) = -gr/R$. Use the chain rule to write
$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}.$$
This gives us the separable equation
$$v\frac{dv}{dr} = -\frac{gr}{R},$$
with the condition $v(R) = 0$. Integrate to obtain
$$v^2 = gR - \frac{gr^2}{R}.$$
Put $r = 0$ to get the speed at the center of the Earth. This is $v = \sqrt{gR} = \sqrt{24} \approx 4.9$ miles per second.

7. Let θ be the angle the chord makes with the vertical. Then
$$m\frac{dv}{dt} = mg\cos(\theta); v(0) = 0.$$
This gives us $s(t) = \frac{1}{2}gt^2\cos(\theta)$, so the time of descent is
$$t = \left(\frac{2s}{g\cos(\theta)}\right)^{1/2},$$
where s is the length of the chord. By the law of cosines, the length of this chord satisfies
$$s^2 = 2R^2 - 2R^2\cos(\pi - 2\theta) = 2R^2(1 + \cos(2\theta)) = 4R^2\cos^2(\theta).$$
Therefore
$$t = 2\sqrt{\frac{R}{g}},$$
and this is independent of θ.

8. The loop currents in Figure 1.13 satisfy the equations
$$10i_1 + 15(i_1 - i_2) = 10$$
$$15(i_2 - i_1) + 30i_2 = 0$$
so
$$i_1 = \frac{1}{2} \text{ amp and } i_2 = \frac{1}{6} \text{ amp.}$$

9. The capacitor charge is modeled by

$$250(10^3)i + \frac{1}{2(10^{-6})}q = 80; q(0) = 0.$$

Put $i = q'$ to obtain, after some simplification,

$$q' + 2q = 32(10^{-5}),$$

a linear equation with solution $q(t) = 16(10^{-5})(1 - e^{-2t})$. The capacitor voltage is

$$E_C = \frac{1}{C}q = 80(1 - e^{-2t}).$$

The voltage reaches 76 volts when $t = (1/2)\ln(20)$, which is approximately 1.498 seconds after the switch is closed. Calculate the current at this time by

$$\frac{1}{2}\ln(20)i = q'(\ln(20)/2) = 32(10^{-5})e^{-\ln(20)} = 16 \text{ micro amps.}$$

10. The loop currents satisfy

$$\begin{aligned} 5(i_1' - i_2') + 10i_2 &= 6, \\ -5i_1' + 5i_2' + 30i_2 + 10(q_2 - q_3) &= 0, \\ -10q_2 + 10q_3 + 15i_3 + \frac{5}{2}q_3 &= 0. \end{aligned}$$

Since $q_1(0+) = q_2(0+) = q_3(0+) = 0$, then from the third equation we have $i_3(0+) = 0$. Add the three equations to obtain

$$10i_1(0+) + 30i_2(0+) = 6.$$

From the upper node between loops 1 and 2, we conclude that $i_1(0+) = i_2(0+)$. Therefore

$$i_1(0+) = i_2(0+) = \frac{3}{20} \text{ amps.}$$

11. (a) Calculate

$$i'(t) = \frac{E}{R}e^{-Rt/L} > 0,$$

implying that the current increases with time.

(b) Note that $(1 - e^{-1}) = 0.63+$, so the inductive time constant is $t_0 = L/R$.

(c) For $i(0) \neq 0$, the time to reach 63 percent of E/R is

$$t_0 = \frac{L}{R}\ln\left(\frac{e(E - Ri(0))}{E}\right),$$

which decreases with $i(0)$.

12. (a) For

$$q' + \frac{1}{RC}q = \frac{E}{R}; q(0) = q_0,$$

the differential equation is linear with integrating factor $e^{t/RC}$. The differential equation becomes

$$(qe^{t/RC})' = \frac{E}{R}e^{t/RC}$$

so

$$q(t) = EC + ke^{-t/RC}.$$

$q(0) = q_0$ gives $k = q_0 - EC$, so

$$q(t) = EC + (q_0 - EC)e^{-T/RC}.$$

(b) $\lim_{t\to\infty} q(t) = EC$, and this independent of q_0.

(c) If $q_0 > EC$, $q_{\text{max}} = q(0) = q_0$, there is no minimum in this case but $q(t)$ decreases toward EC. If $q_0 = EC$, then $q(t) = EC$ for all t. If $q_0 < EC$, $q_{\text{min}} = q(0) = q_0$ and there is no maximum in this case, but $q(t)$ increases toward EC.

(d) To reach 99 percent of the steady-state value, solve

$$EC + (q_0 - EC)e^{-t/RC} = EC(1 \pm 0.01),$$

so

$$t = RC \ln\left(\frac{q_0 - EC}{0.1EC}\right).$$

13. The differential equation of the given family is

$$\frac{dy}{dx} = \frac{4x}{3}.$$

Orthogonal trajectories satisfy

$$\frac{dy}{dx} = -\frac{3}{4x}$$

and are given by

$$y = -\frac{3}{4}\ln|x| + c.$$

14. Differentiate $x + 2y = k$ implicitly to obtain the differential equation $y' = -1/2$ of this family. The orthogonal trajectories satisfy $y' = 2$, and are the graphs of $y = 2x + c$.

15. The differential equation of the family is

$$y' = 2kx = \frac{2x(y-1)}{x^2} = \frac{2(y-1)}{x}.$$

Orthogonal trajectories satisfy $y' = x/2(y-1)$ and are the graphs of the family of ellipses

$$(y-1)^2 + \frac{1}{2}x^2 = c.$$

16. The differential equation of the given family is $dy/dx = -x/2y$. The orthogonal trajectories satisfy $dy/dx = 2y/x$ and are given by $y = cx^2$, a family of parabolas.

17. The differential equation of the given family is found by solving for k and differentiating to obtain $k = \ln(y)/x$, so

$$\frac{dy}{dx} = \frac{y\ln(y)}{x}.$$

Orthogonal trajectories satisfy

$$\frac{dy}{dx} = -\frac{x}{y\ln(y)}.$$

This is separable with solutions

$$y^2(\ln(y^2) - 1) = c - 2x^2.$$

18. At time $t = 0$, assume that the dog is at the origin of an x, y - system and the man is located at $(A, 0)$ on the x - axis. The man moves directly upward into the first quadrant and at time t is at (A, vt). The position of the dog at time $t > 0$ is (x, y) and the dog runs with speed $2v$, always directly toward his master. At time $t > 0$, the man is at (A, vt), the dot is at (x, y), and the tangent to the dog's path joins these two points. Thus

$$\frac{dy}{dx} = \frac{vt - y}{A - x}$$

for $x < A$. To eliminate t from this equation use the fact that during the time the man has moved vt units upward, the dog has run $2vt$ units along his path. Thus

$$2vt = \int_0^x \left[1 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2} d\xi.$$

Use this integral to eliminate the vt term in the original differential equation to obtain

$$2(A - x)y'(x) = \int_0^x \left[1 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2} d\xi - 2y.$$

Differentiate this equation to obtain

$$2(A-x)y''-2y'=(1+(y')^2)^{1/2}-2y',$$

or

$$2(A-x)y''=(1+(y')^2)^{1/2},$$

subject to $y(0)=y'(0)=0$. Let $u=y'$ to obtain the separable equation

$$\frac{1}{\sqrt{1+u^2}}\,du=\frac{1}{2(A-x)}\,dx.$$

This has the solution

$$\ln(u+\sqrt{1+u^2})=-\frac{1}{2}\ln(A-x)+c.$$

Using $y'(0)=u(0)=0$ gives us

$$u+\sqrt{1+u^2}=\frac{\sqrt{A}}{\sqrt{A-x}},$$

or, equivalently,

$$y'+\sqrt{(1+(y')^2)}=\frac{\sqrt{A}}{\sqrt{A-x}};y(0)=0.$$

From the equation for y'', we obtain

$$\sqrt{1+(y')^2}=2(A-x)y'',$$

so

$$y'+2(A-x)y''=\frac{\sqrt{A}}{\sqrt{A-x}};y(0)=y'(0)=0$$

for $x<A$. Let $w=y'$ to obtain the linear first order equation

$$w'+\frac{1}{2(A-x)}w=\frac{\sqrt{A}}{2(A-x)^{3/2}}.$$

An integrating factor is $1/\sqrt{A-x}$ and we can write

$$\frac{d}{dx}\left[\frac{w}{\sqrt{A-x}}\right]=\frac{\sqrt{A}}{2(A-x)^2}.$$

The solution, subject to $w(0)=0$, is

$$w(x)=\frac{A}{\sqrt{2}}\frac{1}{\sqrt{A-x}}-\frac{1}{2\sqrt{A}}\sqrt{A-x}=\frac{dy}{dx}.$$

Integrate one last time to obtain

$$y(x)=-\sqrt{A}\sqrt{A-x}+\frac{1}{3\sqrt{A}}(A-x)^{1/2}+\frac{2}{3}A,$$

in which we have used $y(0) = 0$ to evaluate the constant of integration. The dog catches the man at $x = A$, so they meet at $(A, 2A/3)$. Since this is also (A, vt) when they meet, we conclude that $vt = 2A/3$, so they meet at time

$$t = \frac{2A}{3v}.$$

19. (a) Clearly each bug follows the same curve of pursuit relative to the corner from which it started. Place a polar coordinate system as suggested and determine the pursuit curve for the bug starting at $\theta = 0, r = a/\sqrt{2}$. At any time $t > 0$, the bug will be at $(f(\theta), \theta)$ and its target will be at $(f(\theta), \theta + \pi/2)$, and

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}.$$

On the other hand, the tangent direction must be from $(f(\theta), \theta)$ to $(f(\theta), \theta + \pi/2)$, so

$$\begin{aligned}\frac{dy}{dx} &= \frac{f(\theta)\sin(\theta + \pi/2) - f(\theta)\sin(\theta)}{f(\theta)\cos(\theta + \pi/2) - f(\theta)\cos(\theta)}\\ &= \frac{\cos(\theta) - \sin(\theta)}{-\sin(\theta) - \cos(\theta)}\\ &= \frac{\sin(\theta) - \cos(\theta)}{\sin(\theta) + \cos(\theta)}.\end{aligned}$$

Equate these two expressions for dy/dx and simplify to obtain

$$f'(\theta) + f(\theta) = 0$$

with $f(0) = a/\sqrt{2}$. Then

$$r = f(\theta) = \frac{a}{\sqrt{2}}e^{-\theta}$$

is the polar coordinate equation of the pursuit curve.

(b) The distance traveled by each bug is

$$\begin{aligned}D &= \int_0^\infty \sqrt{(r')^2 + r^2}\, d\theta\\ &= \int_0^\infty \left[\left(\frac{a}{\sqrt{2}}e^{-\theta}\right)^2 + \left(\frac{-a}{\sqrt{2}}e^{-\theta}\right)^2\right]^{1/2} d\theta\\ &= a\int_0^\infty e^{-\theta}\, d\theta = a.\end{aligned}$$

(c) Since $r = f(\theta) = ae^{-\theta}/\sqrt{2} > 0$ for all θ, no bug reaches its quarry. The distance between pursuer and quarry is $ae^{-\theta}$.

20. (a) Assume the disk rotates counterclockwise with angular velocity ω radians per second and the bug steps on the rotating disk at point $(a, 0)$. By the chain rule,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt},$$

so

$$\frac{dr}{d\theta} = -\frac{v}{\omega}.$$

Then

$$r = c - \frac{\theta v}{\omega}, r(0) = a$$

gives us

$$r(\theta) = a - \frac{\theta v}{\omega}.$$

This is a spiral.

(b) To reach the center, solve $r = 0 = a - \theta v/\omega$ to get $\theta = a\omega/v$ radians, or $\theta = a\omega/2\pi v$ revolutions.

(c) The distance traveled is

$$\begin{aligned} s &= \int_0^{a\omega/v} \sqrt{r^2 + (r')^2}\, d\theta \\ &= \int_0^{a\omega/v} \sqrt{\left(a - \frac{v\theta}{\omega}\right)^2 + \left(\frac{v}{\omega}\right)^2}\, d\theta. \end{aligned}$$

To evaluate this integral let $\theta = -z + a\omega/v$, so

$$\begin{aligned} s &= \frac{v}{\omega}\int_0^{a\omega/v} \sqrt{1 + z^2}\, dz \\ &= \frac{1}{2}\left[\frac{a\omega}{v^2}\sqrt{a\omega^2 + v^2} + \ln\left(\frac{a\omega + \sqrt{\omega^2 + v^2}}{v}\right)\right]. \end{aligned}$$

21. Let $x(t)$ denote the length of chain hanging down from the table at time t, and note that once the chain starts moving, all 24 feet move with velocity v. The motion is modeled by

$$\rho x = \frac{24\rho}{g}\frac{dv}{dt} = \frac{3\rho}{4} v \frac{dv}{dx},$$

with $v(6) = 0$. Thus $x^2 = \frac{3}{4}v^2 + c$ and $v(6) = 0$ gives $c = 36$, so

$$v^2 = \frac{4}{3}(x^2 - 36).$$

When the end leaves the table, $x = 24$ so $v = 12\sqrt{5} \approx 26.84$ feet per second. The time is

$$\begin{aligned} t_f &= \int_6^{24} \frac{1}{v(x)}\,dx = \int_6^{24} \frac{\sqrt{3}}{2\sqrt{x^2-36}}\,dx \\ &= \frac{\sqrt{3}}{2}\ln(6+\sqrt{35}) \approx 2.15 \end{aligned}$$

seconds.

22. The force pulling the chain off the table is due to the four feet of chain hanging between the table and the floor. Let $x(t)$ denote the distance the free end of the chain on the table has moved. The motion is modeled by

$$4\rho = \frac{d}{dt}\left[(22-x)\frac{\rho}{g}v\right]; v = 0 \text{ when } x = 0.$$

Rewrite this as

$$128 + v^2 = (22-x)v\frac{dv}{dx},$$

a separable differential equation which we solve to get

$$c - \ln|22-x| = \frac{1}{2}\ln(128+v^2)$$

Since $v = 0$ when $x = 0$, then $c = \ln(176\sqrt{2})$. The end of the chain leaves the table when $x = 18$, so at this time

$$v = \sqrt{3744} \approx 61.19 \text{ feet per second.}$$

1.6 Existence and Uniqueness Questions

1. Both $f(x,y) = \sin(xy)$ and $\partial f/\partial y = x\cos(xy)$ are continuous (for all (x,y)).

2. $f(x,y) = \ln|x-y|$ and

$$\frac{\partial f}{\partial y} = -\frac{1}{x-y}$$

are continuous on a sufficiently small rectangle about $(3,\pi)$, for example, on a square centered at $(3,\pi)$ and having side length $1/100$.

3. Both $f(x,y) = x^2 - y^2 + 8x/y$ and

$$\frac{\partial f}{\partial y} = -2y - \frac{8x}{y^2}$$

are continuous on a sufficiently small rectangle centered at $(3,-1)$, for example, on the square of side length 1.

4. Both $f(x, y) = \cos(e^{xy})$ and $\partial f/\partial y = -xe^{xy}\sin(e^{xy})$ are continuous over the entire plane.

5. By taking $|y'| = y'$, we get $y' = 2y$ and the initial value problem has the solution $y(x) = y_0 e^{2(x-x_0)}$. However, if we take $|y'| = -y'$, then the initial value problem has the solution $y(x) = y_0 e^{-2(x-x_0)}$.

   In this problem we have $|y'| = 2y = f(x, y)$, so we actually have $y' = \pm 2y$, and $f(x, y) = \pm 2y$. This is not even a function, so the terms of Theorem 1.2 do not apply and the theorem offers no conclusion.

6. (a) Since both $f(x, y) = 2-y$ and $\partial f/\partial y = -1$ are continuous everywhere, the initial value problem has a unique solution. In this case the solution is easy to find: $y = 2 - e^{-x}$. This is the answer to (b).

   (c)

$$
\begin{aligned}
&y_0 = 1, y_1 = 1 + \int_0^x dt = 1 + x,\\
&y_2 = 1 + \int_0^x (1-t)\,dt = 1 + x - \frac{x^2}{2},\\
&y_3 = 1 + \int_0^x \left(1 - t + \frac{t^2}{2}\right) dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!},\\
&y_4 = 1 + \int_0^x \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!}\right) dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!},\\
&y_5 = 1 + \int_0^x y_4(t)\,dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!},\\
&y_6 = 1 + \int_0^x y_5(t)\,dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!}.
\end{aligned}
$$

   Based on these computations, we conjecture that

$$
y_n(x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + (-1)^{n+1}\frac{x^n}{n!}
$$

   (d)

$$
\begin{aligned}
2 - e^{-x} &= 2 - \left(1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^n}{n!}\right)\\
&= 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots + (-1)^{n+1}\frac{x^n}{n!} + \cdots
\end{aligned}
$$

   Since

$$
2 - e^{-x} = 2 - \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} = \lim_{n\to\infty} y_n(x),
$$

   the Picard iterates converge to the unique solution of the initial value problem.

7. (a) Since both $f(x,y) = 4 + y$ and $\partial f/\partial y = 1$ are continuous everywhere, the initial value problem has a unique solution.

(b) This linear differential equation is easily solved to yield $y = -4 + 7e^x$ as the unique solution of the initial value problem.

(c)

$$
\begin{aligned}
y_0 &= 3, y_1 = 3 + \int_0^x 7\,dt = 3 + 7x,\\
y_2 &= 3 + \int_0^x (7 + 7t)\,dt = 3 + 7x + 7\frac{x^2}{2},\\
y_3 &= 3 + \int_0^x \left(7 + 7t + 7\frac{t^2}{2}\right)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!},\\
y_4 &= 3 + \int_0^x \left(7 + 7t + 7\frac{t^2}{2} + 7\frac{t^3}{3!}\right)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!},\\
y_5 &= 3 + \int_0^x y_4(t)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!} + 7\frac{x^5}{5!},\\
y_6 &= 3 + \int_0^x y_5(t)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^5}{5!} + 7\frac{x^6}{6!}.
\end{aligned}
$$

(d) We conjecture that

$$y_n(x) = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + \cdots + 7\frac{x^n}{n!}.$$

Note that

$$y_n(x) = -4 + 7\sum_{k=0}^{n} \frac{x^k}{k!}$$

and that

$$\lim_{n\to\infty} y_n(x) = -4 + 7\sum_{k=0}^{\infty} \frac{x^k}{k!} = -4 + 7e^x.$$

Thus the Picard iterates converge to the solution.

8. (a) Both $f(x,y) = 2x^2$ and $\partial f/\partial y = 0$ are continuous everywhere, so the initial value problem has a unique solution.

(b) The solution is

$$y = \frac{2}{3}x^3 + \frac{7}{3}.$$

(c)

$$y_0 = 3, y_1 = 3 + \int_1^x 2t^2\,dt = \frac{2}{3}x^3 + \frac{7}{3}.$$

Because $f(x,y)$ is independent of y, $y_n(x) = y_1(x)$ for all n.

(d) The sequence of Picard iterates is a constant sequence. We can write

$$y = \frac{2}{3}x^3 + \frac{7}{3} = 3 + 2(x-1) + 2(x-1)^2 + \frac{2}{3}(x-1)^3$$

and this is the Taylor expansion of the solution about 1. For $n \geq 3$ the nth partial sum of this finite series is the solution. Certainly $y_n \to y$ as $n \to \infty$.

9. (a) $f(x,y) = \cos(x)$ and $\partial f/\partial y = 0$ are continuous for all (x,y), so the problem has a unique solution.

(b) The solution is $y = 1 + \sin(x)$.

(c)

$$y_0 = 1, y_1 = 1 + \int_{\pi}^{x} \cos(t)\,dt = 1 + \sin(x).$$

In this example, $y_n = y_1$ for $n = 2, 3, \cdots$.

(d) For $n \geq 1$,

$$y = 1 + \sin(x) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}x^{2k+1}}{(2k+1)!}.$$

The nth partial sum T_n of this Taylor series does not agree with the nth Picard iterate $y_n(x)$. However,

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} y_n(x) = 1 + \sin(x),$$

so both sequences converge to the unique solution.