1 COUNTING

1.1 BASIC COUNTING

Pages 7 to 8

Problem 1

Solution

The value of i ranges from 2 to n. When $i = k$, the variable j ranges from 2 to k. Thus, there are at most $k - 1$ comparisons (because we stop if $j = 2$). Thus, the total number of comparisons is

$$1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2} .$$

The algorithm will make this number of comparisons if the original ordering is the reverse of the sorted ordering.

Problem 2

Solution

Number the five teams 1–5. Team 1 must play all four others. Team 2 will be in one of these games but must play in three more games with Teams 3, 4, and 5. Team 3 is in two of the games already mentioned and must still play Teams 4 and 5 for two more games. Team 4 must play Team 5, in addition to playing in three of the games already mentioned. Thus, there are $4 + 3 + 2 + 1 = 10$ games. Alternatively, there are five teams, each of which must play in four games, giving us 20 pairings of two teams each. However, each game involves two of these pairings, so there are $20/2 = 10$ games.

Problem 3

Solution

The set of possible draws is a union of 52 sets (one for each possible first card), each of size 51. So, by the product principle, there are $52 \cdot 51$ ways to draw the two cards.

Problem 4

Solution

The answer is the same as in Problem 3, except we can draw the cards in either order. Therefore, the number of ways is $52 \cdot 51/2 = 1326$.

Problem 5

Solution

$52 \cdot 51 \cdot 50$, by two applications of the product principle.

Problem 6

Solution

$10 \cdot 9 = 90$.

Problem 7

Solution

$\binom{10}{2} = 10 \cdot 9/2 = 45$.

Problem 8

Solution

$10 \cdot \binom{9}{2}$, or $8\binom{10}{2}$.

Problem 9

Solution

This formula counts the number of ways to choose a president and an executive advisory board (not including the president) from a club of n people. The left side chooses the president first, then the committee. The right side chooses the committee first, then the president.

Problem 10

Solution

$m \cdot n$.

Problem 11

Solution

By the product rule, there are $10 \cdot 9 = 90$ ways to choose two-scoop cones with two different flavors. However, according to your mother's rule, the order of scoops doesn't matter. Because each two-scoop cone can be ordered in two different ways (e.g., chocolate over vanilla and vanilla over chocolate), we have $90/2 = 45$ ways of choosing two-scoop cones with different flavors. There are an additional ten cones with the same flavor for both scoops, giving 55 possible cones.

Problem 12

Solution

Because order does matter, we have $10 \cdot 9 = 90$ ways to choose ice cream cones with two distinct flavors, plus ten more with the same flavor for both scoops, giving 100 choices.

Problem 13

Solution

$1 + 2 + 4 + \cdots + 2^{19} = 2^{20} - 1 = 1{,}048{,}575$. Your justification may be neither principle, only the sum principle (the set of all pennies is the union of the set of pennies on Day 1 with those on Day 2, and so on), or both principles (the set of pennies you receive on Day i is the union of two sets of pennies, each of the size that you received on Day $i - 1$). As long as your explanation makes sense, any of these answers is fine.

Problem 14

Solution

$5 \cdot 3 \cdot 3 \cdot 3 = 135$.

Problem 15

Solution

Yes; in Line 4, j could start at $i + 1$ rather than i.

1.2 COUNTING LISTS, PERMUTATIONS, AND SUBSETS

Pages 17 to 19

Problem 1

Solution

For each piece of fruit, we have n choices of who to give it to. So, by version 2 of the product principle, the number of ways to pass out the fruit is n^k.

Problem 2

Solution

$$
\begin{array}{lll}
f_1(1) = a & f_1(2) = a & f_1(3) = a \\
f_2(1) = a & f_2(2) = a & f_2(3) = b \\
f_3(1) = a & f_3(2) = b & f_3(3) = a \\
f_4(1) = a & f_4(2) = b & f_4(3) = b \\
f_5(1) = b & f_5(2) = a & f_5(3) = a \\
f_6(1) = b & f_6(2) = a & f_6(3) = b \\
f_7(1) = b & f_7(2) = b & f_7(3) = a \\
f_8(1) = b & f_8(2) = b & f_8(3) = b
\end{array}
$$

None are one-to-one; all but f_1 and f_8 are onto.