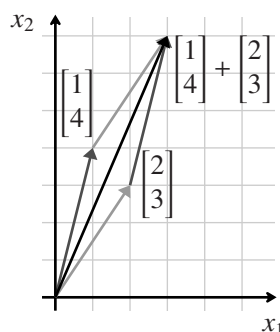


# CHAPTER 1 Euclidean Vector Spaces

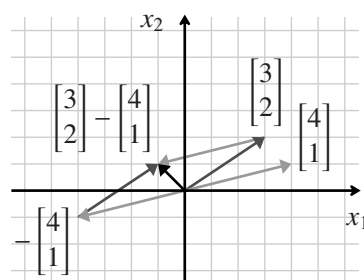
## 1.1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

### Practice Problems

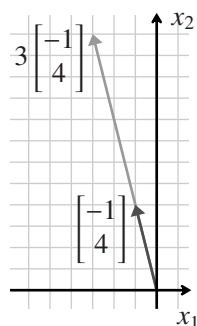
A1 (a)  $\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 4+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$



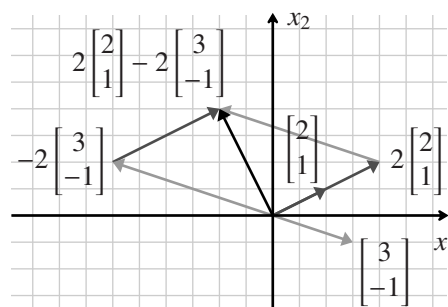
(b)  $\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-4 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



(c)  $3 \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3(-1) \\ 3(4) \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$



(d)  $2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$



A2 (a)  $\begin{bmatrix} 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4+(-1) \\ -2+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(c)  $-2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} (-2)3 \\ (-2)(-2) \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

(e)  $\frac{2}{3} \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1/4 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2/3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$

(b)  $\begin{bmatrix} -3 \\ -4 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -3-(-2) \\ -4-5 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$

(d)  $\frac{1}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 4 \end{bmatrix}$

(f)  $\sqrt{2} \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \end{bmatrix} + 3 \begin{bmatrix} 1 \\ \sqrt{6} \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{6} \end{bmatrix} + \begin{bmatrix} 3 \\ 3\sqrt{6} \end{bmatrix} = \begin{bmatrix} 5 \\ 4\sqrt{6} \end{bmatrix}$

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A3 (a)  $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2-5 \\ 3-1 \\ 4-(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}$

(b)  $\begin{bmatrix} 2 \\ 1 \\ -6 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 2+(-3) \\ 1+1 \\ -6+(-4) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -10 \end{bmatrix}$

(c)  $-6 \begin{bmatrix} 4 \\ -5 \\ -6 \end{bmatrix} = \begin{bmatrix} (-6)4 \\ (-6)(-5) \\ (-6)(-6) \end{bmatrix} = \begin{bmatrix} -24 \\ 30 \\ 36 \end{bmatrix}$

(d)  $-2 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -5 \end{bmatrix}$

(e)  $2 \begin{bmatrix} 2/3 \\ -1/3 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -2/3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -4/3 \\ 13/3 \end{bmatrix}$

(f)  $\sqrt{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \pi \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix} + \begin{bmatrix} -\pi \\ 0 \\ \pi \end{bmatrix} = \begin{bmatrix} \sqrt{2} - \pi \\ \sqrt{2} \\ \sqrt{2} + \pi \end{bmatrix}$

A4 (a)  $2\vec{v} - 3\vec{w} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} - \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ -13 \end{bmatrix}$

(b)  $-3(\vec{v} + 2\vec{w}) + 5\vec{v} = -3 \left( \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \right) + \begin{bmatrix} 5 \\ 10 \\ -10 \end{bmatrix} = -3 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} -15 \\ 0 \\ -12 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ -22 \end{bmatrix}$

(c) We have  $\vec{w} - 2\vec{u} = 3\vec{v}$ , so  $2\vec{u} = \vec{w} - 3\vec{v}$  or  $\vec{u} = \frac{1}{2}(\vec{w} - 3\vec{v})$ . This gives

$$\vec{u} = \frac{1}{2} \left( \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \\ -6 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} -1 \\ -7 \\ 9 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -7/2 \\ 9/2 \end{bmatrix}$$

(d) We have  $\vec{u} - 3\vec{v} = 2\vec{u}$ , so  $\vec{u} = -3\vec{v} = \begin{bmatrix} -3 \\ -6 \\ 6 \end{bmatrix}$ .

A5 (a)  $\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} = \begin{bmatrix} 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 5/2 \\ -1/2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1/2 \end{bmatrix}$

(b)  $2(\vec{v} + \vec{w}) - (2\vec{v} - 3\vec{w}) = 2 \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} - \left( \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 15 \\ -3 \\ -6 \end{bmatrix} \right) = \begin{bmatrix} 16 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} -9 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 25 \\ -5 \\ -10 \end{bmatrix}$

(c) We have  $\vec{w} - \vec{u} = 2\vec{v}$ , so  $\vec{u} = \vec{w} - 2\vec{v}$ . This gives  $\vec{u} = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -4 \end{bmatrix}$ .

(d) We have  $\frac{1}{2}\vec{u} + \frac{1}{3}\vec{v} = \vec{w}$ , so  $\frac{1}{2}\vec{u} = \vec{w} - \frac{1}{3}\vec{v}$ , or  $\vec{u} = 2\vec{w} - \frac{2}{3}\vec{v} = \begin{bmatrix} 10 \\ -2 \\ -4 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 2 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 8 \\ -8/3 \\ -14/3 \end{bmatrix}$ .

A6

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{PS} = \vec{OS} - \vec{OP} = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 4 \end{bmatrix}$$

$$\vec{QR} = \vec{OR} - \vec{OQ} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$$

$$\vec{SR} = \vec{OR} - \vec{OS} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix}$$

Thus,

$$\vec{PQ} + \vec{QR} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix} = \vec{PS} + \vec{SR}$$

A7 (a) The equation of the line is  $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

(b) The equation of the line is  $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ -6 \end{bmatrix}, t \in \mathbb{R}$

(c) The equation of the line is  $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 4 \\ -2 \\ -11 \end{bmatrix}, t \in \mathbb{R}$

(d) The equation of the line is  $\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}$

A8 Note that alternative correct answers are possible.

(a) The direction vector  $\vec{d}$  of the line is given by the directed line segment joining the two points:  $\vec{d} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ . This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \quad t \in \mathbb{R}$$

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- (b) The direction vector  $\vec{d}$  of the line is given by the directed line segment joining the two points:  $\vec{d} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$ . This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} -6 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}$$

- (c) The direction vector  $\vec{d}$  of the line is given by the directed line segment joining the two points:  $\vec{d} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$ . This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}, \quad t \in \mathbb{R}$$

- (d) The direction vector  $\vec{d}$  of the line is given by the directed line segment joining the two points:  $\vec{d} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}$ . This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

- (e) The direction vector  $\vec{d}$  of the line is given by the directed line segment joining the two points:  $\vec{d} = \begin{bmatrix} -1 \\ 1 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 3/4 \\ -2/3 \end{bmatrix}$ . This, along with one of the points, may be used to obtain an equation for the line

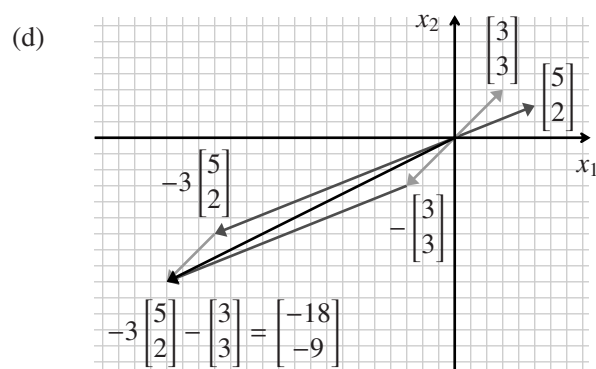
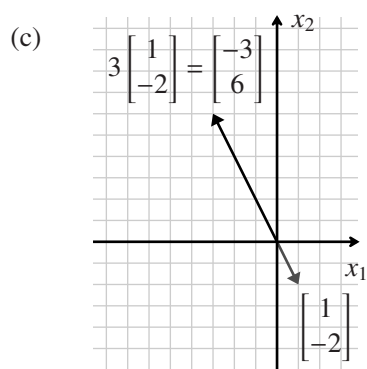
$$\vec{x} = \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3/2 \\ 3/4 \\ -2/3 \end{bmatrix}, \quad t \in \mathbb{R}$$

A9 (a) We have

$$\begin{aligned} x_2 &= 3x_1 + 2 \\ x_2 + 1 &= 3x_1 + 3 \\ x_2 + 1 &= 3(x_1 + 1) \end{aligned}$$



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**B2** (a)  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  (b)  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  (c)  $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$  (d)  $\begin{bmatrix} 0 \\ 27/5 \end{bmatrix}$  (e)  $\begin{bmatrix} 2\sqrt{3} \\ \sqrt{2} - 3\sqrt{3}/2 \end{bmatrix}$

**B3** (a)  $\begin{bmatrix} 3 \\ -3 \\ -6 \end{bmatrix}$  (b)  $\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$  (c)  $\begin{bmatrix} 8 \\ -20 \\ -4 \end{bmatrix}$  (d)  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  (e)  $\begin{bmatrix} 7/3 \\ -4/3 \\ 13/3 \end{bmatrix}$  (f)  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**B4** (a)  $\begin{bmatrix} 21 \\ 4 \\ -5 \end{bmatrix}$  (b)  $\begin{bmatrix} -1 \\ -12 \\ 1 \end{bmatrix}$  (c)  $\begin{bmatrix} -7 \\ -13/2 \\ 2 \end{bmatrix}$  (d)  $\begin{bmatrix} 9 \\ -1/2 \\ -2 \end{bmatrix}$

**B5** (a)  $\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$  (b)  $\begin{bmatrix} 0 \\ 1/4 \\ -3/4 \end{bmatrix}$  (c)  $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  (d)  $\vec{u} = \begin{bmatrix} 4 \\ -7/2 \\ 5/2 \end{bmatrix}$

**B6** (a)  $\vec{PQ} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ ,  $\vec{PR} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ ,  $\vec{PS} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}$ ,  $\vec{QR} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$ ,  $\vec{SR} = \begin{bmatrix} -9 \\ -2 \\ 7 \end{bmatrix}$

(b)  $\vec{PQ} = \begin{bmatrix} -1 \\ 9 \\ -4 \end{bmatrix}$ ,  $\vec{PR} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$ ,  $\vec{PS} = \begin{bmatrix} -5 \\ 6 \\ -2 \end{bmatrix}$ ,  $\vec{QR} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix}$ ,  $\vec{SR} = \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}$

**B7** (a)  $\vec{x} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ ,  $t \in \mathbb{R}$ . (b)  $\vec{x} = t \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ ,  $t \in \mathbb{R}$ .

(c)  $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$ ,  $t \in \mathbb{R}$ . (d)  $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ ,  $t \in \mathbb{R}$ .

(a)  $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ,  $t \in \mathbb{R}$  (b)  $\vec{x} = t \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ ,  $t \in \mathbb{R}$

**B8** (c)  $\vec{x} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 11 \\ -1 \end{bmatrix}$ ,  $t \in \mathbb{R}$  (d)  $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1/2 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 4/3 \\ 1/2 \end{bmatrix}$ ,  $t \in \mathbb{R}$

**B9** Other correct answers are possible.

$$(a) \quad x_1 = -\frac{1}{2}t + \frac{3}{2}, x_2 = t; \vec{x} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$(b) \quad x_1 = t, x_2 = -\frac{1}{2}t + \frac{3}{2}; \vec{x} = \begin{bmatrix} 0 \\ 3/2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$$

**B10** (a) Since  $-2\vec{PQ} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} = \vec{PR}$ , the point  $P$ ,  $Q$ , and  $R$  must be collinear.

(b) Since  $-\vec{ST} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix} = \vec{SU}$ , the point  $S$ ,  $T$ , and  $U$  must be collinear.

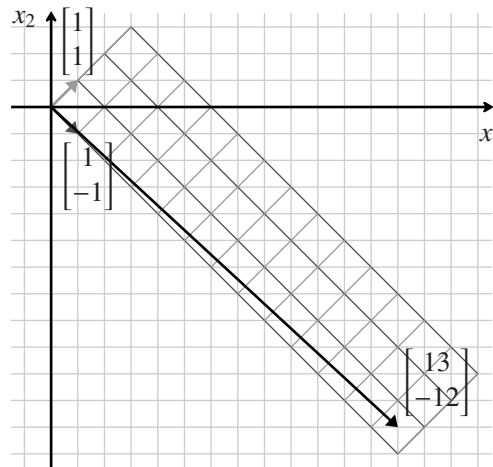
### Computer Problems

**C1** (a)  $\begin{bmatrix} -2322 \\ -1761 \\ 1667 \end{bmatrix}$  (b)  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

### Conceptual Problems

**D1** (a) We need to find  $t_1$  and  $t_2$  such that  $\begin{bmatrix} 13 \\ -12 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ t_1 - t_2 \end{bmatrix}$ .

That is, we need to solve the two equations in two unknowns  $t_1 + t_2 = 13$  and  $t_1 - t_2 = -12$ . Using substitution and/or elimination we find that  $t_1 = \frac{1}{2}$  and  $t_2 = \frac{25}{2}$ .



(b) We use the same approach as in part (a). We need to find  $t_1$  and  $t_2$  such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ t_1 - t_2 \end{bmatrix}$$

Solving  $t_1 + t_2 = x_1$  and  $t_1 - t_2 = x_2$  by substitution and/or elimination gives  $t_1 = \frac{1}{2}(x_1 + x_2)$  and  $t_2 = \frac{1}{2}(x_1 - x_2)$ .

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(c) We have  $x_1 = \sqrt{2}$  and  $x_2 = \pi$ , so we get  $t_1 = \frac{1}{2}(\sqrt{2} + \pi)$  and  $t_2 = \frac{1}{2}(\sqrt{2} - \pi)$ .

**D2** (a)  $\vec{PQ} + \vec{QR} + \vec{RP}$  can be described informally as “start at  $P$  and move to  $Q$ , then move from  $Q$  to  $R$ , then from  $R$  to  $P$ ; the net result is a zero change in position.”

(b) We have  $\vec{PQ} = \vec{q} - \vec{p}$ ,  $\vec{QR} = \vec{r} - \vec{q}$ , and  $\vec{RP} = \vec{p} - \vec{r}$ . Thus,

$$\vec{PQ} + \vec{QR} + \vec{RP} = \vec{q} - \vec{p} + \vec{r} - \vec{q} + \vec{p} - \vec{r} = \vec{0}$$

**D3** Assume that  $\vec{x} = \vec{p} + t\vec{d}$ ,  $t \in \mathbb{R}$ , is a line in  $\mathbb{R}^2$  passing through the origin. Then, there exists a real number  $t_1$  such that  $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{p} + t_1\vec{d}$ . Hence,  $\vec{p} = -t_1\vec{d}$  and so  $\vec{p}$  is a scalar multiple of  $\vec{d}$ . On the other hand, assume that  $\vec{p}$  is a scalar multiple of  $\vec{d}$ . Then, there exists a real number  $t_1$  such that  $\vec{p} = t_1\vec{d}$ . Hence, if we take  $t = -t_1$ , we get that the line with vector equation  $\vec{x} = \vec{p} + t\vec{d}$  passes through the point  $\vec{p} + (-t_1)\vec{d} = t_1\vec{d} - t_1\vec{d} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  as required.

**D4** Let  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ . Then,

$$t(\vec{x} + \vec{y}) = t \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} = \begin{bmatrix} t(x_1 + y_1) \\ t(x_2 + y_2) \\ t(x_3 + y_3) \end{bmatrix} = \begin{bmatrix} tx_1 + ty_1 \\ tx_2 + ty_2 \\ tx_3 + ty_3 \end{bmatrix} = t \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + t \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = t \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + t \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = t\vec{x} + t\vec{y}$$

## 1.2 Vectors in $\mathbb{R}^n$

### Practice Problems

**A1** (a)  $\begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \\ 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 1 \\ -2 \\ 5 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ 3 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ 10 \\ -5 \end{bmatrix}$

(c)  $2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 2 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 3 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

**A2** (a) Since the condition of the set contains the square of a variable in it, we suspect that it is not a subspace. To prove it is not a subspace we just need to find one example where the set is not closed under addition.



Let  $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ . Observe that  $\vec{x}$  and  $\vec{y}$  are in the set since  $x_1^2 - x_2^2 = 1^2 - 1^2 = 0 = x_3$  and

$y_1^2 - y_2^2 = 2^2 - 1^2 = 3 = y_3$ , but  $\vec{x} + \vec{y} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$  is not in the set since  $3^2 - 2^2 = 5 \neq 3$ .

- (b) Since the condition of the set only contains linear variables, we suspect that this is a subspace. To prove it is a subspace we need to show that it satisfies the definition of a subspace.

Call the set  $S$ . First, observe that  $S$  is a subset of  $\mathbb{R}^3$  and is non-empty since the zero vector satisfies the

conditions of the set. Pick any vectors  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$  in  $S$ . Then they must satisfy the condition of

$S$ , so  $x_1 = x_3$  and  $y_1 = y_3$ . We now need to show that  $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$  satisfies the conditions of the set.

In particular, we need to show that the first entry of  $\vec{x} + \vec{y}$  equals its third entry. Since  $x_1 = x_3$  and  $y_1 = y_3$  we get  $x_1 + y_1 = x_3 + y_3$  as required. Thus,  $S$  is closed under addition. Similarly, to show  $S$  is closed under

scalar multiplication, we let  $t$  be any real number and show that  $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \end{bmatrix}$  satisfies the conditions of the

set. Using  $x_1 = x_3$  we get  $tx_1 = tx_3$  as required. Thus,  $S$  is a subspace of  $\mathbb{R}^3$ .

- (c) Since the condition of the set only contains linear variables, we suspect that this is a subspace. Call the set  $S$ . First, observe that  $S$  is a subset of  $\mathbb{R}^2$  and is non-empty since the zero vector satisfies the

conditions of the set. Pick any vectors  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$  in  $S$ . Then they must satisfy the condition

of  $S$ , so  $x_1 + x_2 = 0$  and  $y_1 + y_2 = 0$ . Then  $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$  satisfies the conditions of the set since

$(x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = 0 + 0 = 0$ . Thus,  $S$  is closed under addition. Similarly, for any

real number  $t$  we have that  $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix}$  and  $tx_1 + tx_2 = t(x_1 + x_2) = t(0) = 0$ , so  $S$  is also closed under scalar

multiplication. Thus,  $S$  is a subspace of  $\mathbb{R}^2$ .

- (d) The condition of the set involves multiplication of entries, so we suspect that it is not a subspace. Observe

that if  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , then  $\vec{x}$  is in the set since  $x_1x_2 = 1(1) = 1 = x_3$ , but  $2\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$  is not in the set since

$2(2) = 4 \neq 2$ . Therefore, the set is not a subspace.

- (e) At first glance this might not seem like a subspace since we are adding the vector  $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ . However, the key

observation to make is that  $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$  is equal to  $1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . In particular, this is the equation of a plane in  $\mathbb{R}^3$

which passes through the origin. So, this should be a subspace of  $\mathbb{R}^3$ . We could use the definition of a subspace to prove this, but the point of proving theorems is to make problems easier. Therefore, we instead

observe that this is a vector equation of the set  $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$  and hence is a subspace by Theorem 1.2.2.

(f) The set is a subspace of  $\mathbb{R}^4$  by Theorem 1.2.2.

A3 (a) Since the condition of the set only contains linear variables, we suspect that this is a subspace. Call the set  $S$ . By definition  $S$  is a subset of  $\mathbb{R}^4$  and is non-empty since the zero vector satisfies the conditions of the

set. Pick any vectors  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$  in  $S$ , then  $x_1 + x_2 + x_3 + x_4 = 0$  and  $y_1 + y_2 + y_3 + y_4 = 0$ . We

have  $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix}$  satisfies the conditions of the set since  $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) + (x_4 + y_4) =$

$x_1 + x_2 + x_3 + x_4 + y_1 + y_2 + y_3 + y_4 = 0 + 0 = 0$ . Thus,  $S$  is closed under addition. Similarly, for any real

number  $t$  we have that  $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \\ tx_4 \end{bmatrix}$  and  $tx_1 + tx_2 + tx_3 + tx_4 = t(x_1 + x_2 + x_3 + x_4) = t(0) = 0$ , so  $S$  is also

closed under scalar multiplication. Thus,  $S$  is a subspace of  $\mathbb{R}^4$ .

(b) The set clearly does not contain the zero vector and hence cannot be a subspace.

(c) The conditions of the set only contain linear variables, but we notice that the first equation  $x_1 + 2x_3 = 5$  excludes  $x_1 = x_3 = 0$ . Hence the zero vector is not in the set so it is not a subspace.

(d) The conditions of the set involve a multiplication of variables, so we suspect that it is not a subspace. Using

the knowledge gained from problem A2(d) we take  $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ . Then,  $\vec{x}$  is in the set since  $x_1 = 1 = 1(1) = x_3x_4$

and  $x_2 - x_4 = 1 - 1 = 0$ . But,  $2\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$  is not in the set since  $2 \neq 2(2)$ .

(e) Since the conditions of the set only contains linear variables, we suspect that this is a subspace. Call the set  $S$ . By definition  $S$  is a subset of  $\mathbb{R}^4$  and is non-empty since the zero vector satisfies the conditions

of the set. Pick any vectors  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$  in  $S$ , then  $2x_1 = 3x_4$ ,  $x_2 - 5x_3 = 0$ ,  $2y_1 = 3y_4$ , and

$y_2 - 5y_3 = 0$ . We have  $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix}$  satisfies the conditions of the set since  $2(x_1 + y_1) = 2x_1 + 2y_1 =$

$3x_4 + 3y_4 = 3(x_4 + y_4)$  and  $(x_2 + y_2) - 5(x_3 + y_3) = x_2 - 5x_3 + y_2 - 5y_3 = 0 + 0 = 0$ . Thus,  $S$  is closed under